Simple Newsvendor Heuristics for Multiechelon Distribution Networks

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May 5, 2005

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We consider the problem of determining optimal stocking levels in a multi-echelon distribution network consisting of m echelons and n non-identical terminal locations. Lead-times are deterministic, there are no fixed ordering costs, and unmet demand is backlogged. Both Clark and Scarf (1960) and Federgruen and Zipkin (1984b) propose heuristic solutions for such a problem based on a stochastic dynamic programming formulation. The disadvantage of their formulations lies in the very large state space needed for its solution. For serial supply chains, Shang and Song (2003) provide single period newsvendor problems that bound the optimal stocking levels determined by the Clark and Scarf (1960) serial supply chain model. Newsvendor bounds have a number of valuable qualities; they are considerably less computationally intensive, allow for ready parametric analysis, and facilitate the development of intuition. In this paper, we extend the newsvendor bounds technique to distribution systems, thus providing a simple and surprisingly accurate heuristic. Through a simulation study, we show that our heuristic significantly outperforms other common heuristics over a wide range of parameter values. The closed form solutions provided by the newsvendor bounds also allow insights into the system behavior of a distribution network that was not previously possible through alternative solution techniques.

1. Introduction

We consider the problem of determining optimal stocking levels in a multi-echelon distribution network consisting of *m* echelons and *n* non-identical terminal locations. Inventory stocking levels are chosen and controlled by a central decision maker and inventory is monitored on a periodic basis. Optimal solutions of this problem are problematic because of the allocation policy at the branched locations. Both Clark and Scarf (1960) and Federgruen and Zipkin (1984b) propose heuristic solutions for this problem based on a stochastic dynamic programming formulation. The disadvantage of such a formulation lies in the very large state space needed for its solution, thus several simpler heuristics have since been proposed (e.g. Jackson 1988, McGavin et al. 1993, Graves 1996, and Axsater et al. 2002). All of these heuristics face the trade-off of performance and complexity and no rigorous comparison of them exists.

For serial supply chains, Shang and Song (2003) provide a series of single period newsvendor problems, the solution to which bound the optimal stocking levels as determined by Clark and Scarf. Newsvendor bounds have a number of valuable qualities; they are considerably less computationally intensive, allow for ready parametric analysis, and facilitate the development of intuition. In this paper, we extend the newsvendor bounds technique to distribution systems and show that it outperforms other proposed heuristics in both simplicity and performance.

Traditional depictions of two-echelon, single warehouse systems focus on minimizing the total supply chain costs by determining inventory stocking levels for each installation and applying an allocation policy for the warehouse to utilize when it cannot fill all retailer demands. Because of the large dimensionality of the resulting dynamic program, a common approach is to approximate the system and conduct a recursive search over stocking levels. Our newsvendor heuristic avoids such a search, requiring only the solution of a set of simple closed form functions to set base stock levels. The heuristic works as follows. We assume linear echelon holding costs are assessed as well as linear backordering costs for unmet demand at the retail stages but ordering cost are negligible, thus base-stock policies are optimal for each installation. Demands at the retailer stages are assumed to be independent and any unmet demand is fully backordered. Given these assumptions, we bound the costs and base-stock levels of the arborescent system by a single serial system on the low side and a set of n decomposed serial chains on the high side. After solving for the base-stock levels of the resulting serial systems, we take the average of the resulting system wide base stock levels as our heuristic for the original arborescent system.

Due to the unavailability of practical analytical solution methods, we test our heuristic through an extensive and rigorous simulation experiment and compare its performance against other common heuristics. We find that our approach results in an average difference in costs from an optimal solution of 0.57% for symmetric retailers and 0.87% for asymmetric retailers, and 0.62% and 0.75% for systems with 2 or 4 retailers, respectively, easily outperforming all other tested heuristics. Our closed form solution also allows us to generate insights on the effects of altering system parameters on the stocking levels and system costs. For example, we show how increasing asymmetry in costs and demand among the retailers leads to lower stocking levels and lower total costs, but the echelons in which the costs are incurred shift. Finally, we suggest applications of the approach in a variety of related contexts, such as non-arborescent supply chains, delayed differentiation, and network design problems.

2. Literature Review

Two main challenges exist in determining optimal supply chain strategies for distribution systems: determining the stocking policies for each installation and the allocation policy of inventory to downstream stages when demand exceeds supply at the upstream stage. Prior work on these elements of the problem is discussed in §2.1 and §2.2 below.

2.1. Allocation Policies

Clark and Scarf (1960) inspired a long stream of literature on the domain of production and distribution networks with their analysis of serial systems, finding that for systems with a single retailer, echelon inventory stocking policies are optimal. They further suggest that arborescent systems may be approximated by a serial system under a balance allocation assumption, a relaxation of the traditional dynamic program formulation, where the warehouse may reallocate downstream inventory by imposing negative inventory shipments on downstream installations. This approach is utilized frequently in this literature (e.g. see Eppen and Schrage 1981, Federgruen and Zipkin 1984a, 1984b, Federgruen 1993, Verrijdt and de Kok 1996, Garg and Tang 1997, van der Heijden et al. 1997) although in practice such a policy may not always be feasible. Eppen and Schrage (1981) and Erkip et al. (1990) provide simulation results suggesting that for high service level systems, such an allocation policy is feasible most of the time. Unfortunately, the balance relaxation may be inappropriate when downstream installations are substantially asymmetric in inventory cost profiles and lead times (e.g. see Clark and Scarf 1960, Federgruen and Zipkin 1984a, McGavin et al. 1997, and Axsater et al. 2002). Additionally, such an assumption is unrealistic in practice, as it implies the existence of costless and instantaneous transshipment opportunities. Recognizing that while the balance relaxation serves to make the problem tractable, it is also severely limiting in addressing complex and realistic distribution systems, hence we avoid the use of such a relaxation in this work.

A number of allocation policies that do not rely on the balance relaxation have already been introduced in the literature. Graves (1996) utilizes a virtual assignment rule, where echelon inventory is devoted to a given retailer as demand occurs. This is essentially the opposite of the rebalancing assumption where rather than assigning inventory at the end of the supply chain, the assignment occurs before the inventory even enters the system. We use both of these allocation policies to create lower and upper bounds on system stock, respectively.

Erkip (1984), Jackson and Muckstadt (1989), McGavin et al. (1993) and Axsater et al. (2002) consider policies where retailers order from the warehouse two times during the warehouse's order cycle. These models show that splitting an arriving order at the warehouse into two quantities, one of which is shipped immediately and the other at some period before the next arrival of inventory at the warehouse, captures most of the risk pooling benefits. In periodic review systems such as are considered here, the warehouse may ship inventory in every period. That is, we do not restrict the warehouse to two shipment opportunities per order cycle.

By using a random allocation policy, Cachon (2001) develops exact results for the retailer and warehouse costs, although such a policy does not consider the relative need for inventory at the retailers and is hence clearly sub-optimal. Myopic allocation policies, used in our heuristic and by Federgruen and Zipkin (1984b) and Axsater et al (2002), allocate inventory such that the expected costs at the retailers are minimized in the period the inventory arrives (i.e. after the warehouse to retailer shipment lead-time). Federgruen and Zipkin (1984b) show that, for identical retailers, these myopic policies are approximately optimal under general cost structures when orders may be placed every period. Jackson and Muckstadt (1989) and Jackson (1988) use a similar allocation rule, denoted the "runout allocation rule", where the allocation is determined by solving an optimization problem over the horizon until the next arrival of inventory at the warehouse stage. The allocation rule used in this paper is most similar to that of McGavin et al. (1993), who assume identical retailers and allocate stock so as to maximize the minimum retailer inventory position. In contrast, we allow for non-identical downstream stages and minimize the maximum deviation between each installation's echelon inventory-transit position and its echelon base-stock level. Because our base-stock levels are determined by a myopic newsvendor problem, our allocation policy is also a myopic policy.

2.2 Stocking Policies

The traditional approach used in determining stocking levels for a distribution system is to formulate the problem as a stochastic dynamic program and apply relaxations or restrictions to the system to allow for tractability. Federgruen (1993) notes that no efficient algorithms for determining optimal stocking levels exist for most deterministic demand systems. This situation is exacerbated by models with stochastic demand. One particularly tricky issue, for example, is that in the absence of the balancing relaxations, the optimal stock at an installation may be less than the actual stock on hand, given the state of other installations (e.g. consider two retailers, one with ample safety stock and the other carrying backorders). The resulting large solution space for the optimal policy is accompanied by considerable computational burden, hence researchers tend to use approximations and compare heuristic policies via numerical solutions or simulation to known bounds or the "best found system" (McGavin et al. 1993).

Because the literature in this area typically utilizes two-echelon models with a single warehouse and multiple retailers, we begin with a survey of models of this type. One approach is to treat the warehouse as a cross-docking facility that may not hold inventory (e.g. Eppen and Schrage 1981, who determine the average inventory and backorder levels assuming identical retailers and independent demands). Erkip et al. (1990) extend this model to allow for correlated demands and Garg and Tang (1997) extend it to an arbitrary number of echelons and retailers (assuming arborescence holds). Unlike these works, we allow inventory to be held at all upstream locations, thus protecting the retailers from uncertainty in demand over the lead-time from the supplier to the warehouse, in addition to exploiting potential holding cost savings at the warehouse.

When inventory is held at the warehouse, Federgruen and Zipkin (1984a) show that the cost function of a two-echelon distribution system with identical retailers may be approximated by relaxing the dynamic programming formulation to allow rebalancing between the retailers. This rebalancing assumption provides a lower bound to a system's cost, and corresponding stocking levels, when physical transshipments among the retailers are not possible. Their proposed policy is essentially the same as a decentralized system where each installation follows its' own critical number policy. Chen and Zheng (1994) provide lower bounds for the total inventory related costs for such a system after noting that optimal policies are unknown. We adopt a similar bounding approach for our lower bound of an arbitrary arborescent multiechelon system.

Jackson (1988) provides an extension of the Eppen and Schrage (1981) model to allow the warehouse to hold inventory when the warehouse orders every M periods. By allowing inventory to be held at the warehouse, he captures the "depot effect" of allocating inventory to retailers late in the warehouse's order cycle, creating more balanced inventory positions (and hence service levels). Jackson proposes base-stock policies with allocations to the retailers in each period that the warehouse holds sufficient inventory to fill all orders, providing for a finer degree of control than the single mid-cycle allocations of Erkip (1984), Jonsson and Silver (1987 a, b), and Jackson and Muckstadt (1989). Jackson defines a cost function over a single warehouse order horizon of M periods, and sets retailer order-up-to levels based on an approximate problem. Our approach is similar to that of Jackson in that our stocking policy is a function of a sum of newsvendor cost functions, but the decision only involves the quantity to hold at the warehouse in the beginning of the warehouse order cycle as opposed to Jackson's nested optimization problem. Thus while Jacksons' approximate cost function is minimized by searching over a single variable, our heuristic does not require recursive solutions.

Axsater et al. (2002) consider a two-echelon multiple retailer distribution system with no ordering costs but where the orders occur in batches, and the warehouse orders in multiples of a batch size (a system-batch). They propose heuristics to avoid the computationally impractical solution of the stochastic dynamic programming problem. Their virtual assignment heuristic determines stocking levels and is related to Graves (1996) by decomposing the system into multiple independent serial distributor-retailer systems. Future reallocation at the warehouse stage is permitted, but the warehouse orders as if it must fill each retailer's order separately. They argue that this treatment creates an upper bound on stocking levels and costs. We apply the same technique to determine our upper bound.

Thus far we have discussed a number of works where the solution technique has been to relax or constrain the problem to establish tractability. Cachon (2001) also considers a periodic review system with batch ordering, but provides for exact results that may be obtained through a recursive process. In contrast to the works cited above, Cachon utilizes a random allocation policy. This allows for an exact expression to be developed for each retailer's lead-time distribution, thereby providing exact results for average inventory and backorder levels for a given stocking policy. Cachon uses a bounded iterative search to determine stocking levels, and finds that other simple and commonly used heuristics fail to reliably perform well. We confirm these findings while introducing a simple closed form heuristic that does perform well. We also show that as the allocation policy becomes more sophisticated, shifting from random to myopic allocations, the results of our approach outperform all other methods. These results suggest that, like the use of the rebalancing assumption, the use of random allocation policies improves tractability but at the cost of decreased performance. Furthermore, no performance testing exits for such an allocation policy on more realistic systems.

Although many authors note that their approach may be extended beyond two-echelon distribution systems, (e.g. Graves, 1996, Cachon, 2001), the literature is sparse with analytical, numeric, or simulation results for generic arborescent topologies. Notable exceptions include Federgruen and Zipkin (1984b), Garg and Tang (1997), and van der Heijden et al. (1997). Of these, only van der Heijden et al. allow inventory to be held at the non-retail stages. Our approach may be trivially extended to any number of arborescent echelons with non-identical retailers, and inventory may be held at every level in the supply chain.

3. Model

Consider a multi-echelon supply chain with a single supplier of an abundantly available commodity. Label the terminal stages of the system as 1_α where $\alpha = 1, ..., n$, and the furthest upstream stage as m, to denote an m-echelon, n-retailer system. Label intermediate stages beginning with the installation just upstream of a retailer as 2_α, 3_α, …, m_α. The simplest example of such a system has a single distribution point and n retailers, as depicted in Figure 1.

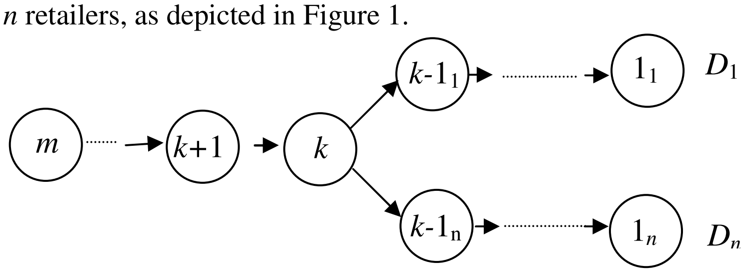

Figure 1: Model of Supply Chain Network

We assume a periodic review policy where the system updates as follows: Allow L_i to represent the known, deterministic lead-time of shipments from the $i+1^{th}$ to the i^{th} installation. At the beginning of period t, a shipment arrives to installation i that was shipped L_i time units ago from installation $i+1$. Existing backorders are satisfied and demand (D_1, ..,. D_n) occurs at the final stages of the echelon tree, nodes 1_1 to 1_n. Unmet demand is backordered. A centralized decision maker considers the current state of inventory in the chain and places an order, Y^t_i, for each installation. The upstream installation consequently ships a quantity, Z^t_{i+1}, that is the minimum of the order quantity at stage $i+1$ and the shipment allocation to stage i (note that for the serial case, the shipment allocation is equal to the available inventory at stage $i+1$).

Our allocation policy requires us to define a "shortage distance" as the difference between the echelon base-stock level and the echelon inventory-transit position (defined as the echelon net inventory plus the inventory in transit to the installation) for each installation. We allocate inventory to minimize the maximum shortage distance. This ensures that if the base-stock levels are appropriately set, then installations receive inventory on the basis of their relative need.

Costs for the system are assessed as follows. After the demand for each period has occurred, but before orders are shipped, a linear echelon holding cost is assessed for the echelon inventory level of each location. Linear backordering costs are assessed for each terminal stage. Both backordering and echelon holding cost rates may differ for different installations at a given echelon; in which case we refer to the cost structure as asymmetric. The stocking policy used in this work is the solution to a set of 2 $(n+1)$ m newsvendor problems, where each newsvendor problem arises due to the decomposition described in §4.

4. Branched Multi-Echelon Newsvendor Heuristic

In this section, we present a heuristic for determining echelon base-stock levels for branched supply chains. We construct two sets of serial supply chain systems that bound the optimal costs and base-stock levels of a branched chain from above and below. For simplicity of exposition, we utilize an m-echelon, n-retailer system with a single distribution point, k, and a symmetric cost structure, but we emphasize that the approach holds for any arborescent distribution network. Our illustrative network is depicted in Figure 1, and faces demand processes D_1, D_2, ... , D_n at the terminal ends of the chain.

To determine the upper bound, we restrict the installations at and upstream of node k to designate and maintain product specific inventories. That is, the centralized decision maker must devote each incoming unit of inventory as it enters the supply chain at installation m to a particular terminal installation. In spirit, this is similar to the virtual assignment approach of Graves (1996), who notes that because it may be desirable to un-commit stock at some downstream stage, this assignment rule will not perform as well as a dynamic allocation policy. This restriction decomposes the arborescent distribution network into a set of n independent serial systems, one system for each terminal retailer, as depicted in Figure 2.

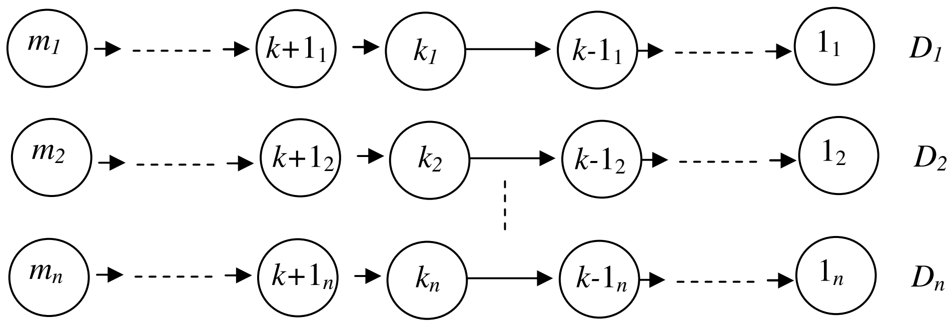

Figure 2: Decomposed Serial Chains

To describe our heuristic, we need the following notation. Where possible throughout this paper, we suppress subscripts and superscripts when the context is clear.

b_α = backorder cost at terminal stage 1_α

$h_{i,\alpha}$ = echelon holding cost rate for installation i in serial chain α

$H_{i,\alpha}$ = local holding cost rate for installation i in serial chain α, equal to $\sum_{j=i}^{m} h_{j\alpha}$

μ_α = mean demand rate at terminal stage 1_α, equal to $E[D_\alpha]$

$L_{i,\alpha}$ = lead-time from stage i +1 to stage i in serial chain α

$\tilde{L}_{i,\alpha}$ = downstream cumulative lead-time equal to $\sum_{j=1}^{i} L_{i,\alpha}$

D^i_α = lead-time demand for stage i in serial chain α

$(x)^-$ = $\max\{0, -x\}$

$s_{i,\alpha}$ = a base-stock level for installation i in serial chain α

$\boldsymbol{s}_i$ = a set of base-stock policies $(s_{i1}, s_{i2},\ldots, s_{in})$

$s_{i,\alpha}^*$ = the optimal base-stock level for installation i in serial chain α

$\boldsymbol{s}_i^*$ = the set of optimal base-stock policies $(s_{i1}^*, s_{i2}^*,\ldots, s_{in}^*)$

$C^s_{i,\alpha}(\boldsymbol{s}_i)$ = the expected per period cost of the first i installations of serial chain α under base-stock policy $\boldsymbol{s}_i$

$C^s_{m,\alpha}(\boldsymbol{s}_i^*)$ = the optimal expected per period cost of serial chain α, equivalently $C_{m\alpha}^*$, $C_{m\alpha}(s_{i\alpha}^*)$

$\boldsymbol{s}^{o*}$ = the optimal base-stock policy for the arborescent network

$C^o(\boldsymbol{s}^{o*})$ = the expected per period cost of the arborescent network under base-stock policy $\boldsymbol{s}^{o*}$

u, l = superscripts denoting upper and lower bounds, respectively

d, c = superscripts denoting decomposed and collapsed systems, respectively

a = superscript denoting the heuristic policy

$s_{i,\alpha}^{u,d}$ = the upper bound of the echelon i base stock level of the decomposed system sub-chain α.

$s_{i,\alpha}^{l,d}$ = the lower bound of the echelon i base stock level of the decomposed system sub-chain α.

$s_i^{u,c}$ = the upper bound of the echelon i base stock level of the collapsed system.

$s_i^{l,c}$ = the upper bound of the echelon i base stock level of the collapsed system.

$C_{i,\alpha}^{l,d}\left(s_{i,\alpha}^{u,d}\right)$ = the lower bound of the expected per period cost of the decomposed system sub-chain α.

$C_{i,\alpha}^{u,d}\left(s_{i,\alpha}^{l,d}\right)$ = the upper bound of the expected per period cost of the decomposed system sub-chain α.

$C_i^{l,c}\left(s_i^{u,c}\right)$ = the lower bound of the expected per period cost of the collapsed system.

$C_i^{u,d}\left(s_i^{l,d}\right)$ = the upper bound of the expected per period cost of the collapsed system.

The optimal base-stock policy of each of the independent serial systems may be determined as follows. Let $C_{0,\alpha} = b_\alpha + H_{\alpha,1}(\mathrm{x})^-$ and $s_{i,0}{}^* = \infty$. For $i = 1, 2, ..., m$, solve the recursive optimization equations

$$C_{i,\alpha}^s(y) = \mathrm{E}\left[h_{i,\alpha}\left(y - D^i{}_\alpha\right) + C_{i-1,\alpha}^s\left(\min\left\{s_{i-1,\alpha}^*, y - D^i{}_\alpha\right\}\right)\right] \tag{1}$$

where $s_{i\alpha}^* = \arg\min\left\{C^s{}_{i,\alpha}\left(y\right)\right\}$. (2)

The optimal expected cost for each decomposed system is

$$C_{m,\alpha}^d\left(s_{m,\alpha}^*\right) = C_\alpha^d\left(s_\alpha^*\right) \tag{3}$$

and the expected overall cost of the total system of independent systems is simply the sum

$$\sum_{\alpha=1}^{n} C_\alpha^d\left(s_\alpha^*\right). \tag{4}$$

Because this sum is obtained by applying an additional constraint to the common installations, it is an upper bound for the optimal cost of the branched network. Additionally, removing the decomposition constraint allows for risk pooling, hence if we assume that backordering costs are sufficiently high to induce installations to carry positive safety stock, the sums

$$\sum_{\alpha} s^{*}_{i,\alpha} \text{ for } i = k, \ldots, m \tag{5}$$

provide an upper bound for base-stock levels for the common installations. Gallego et al. (2003) make a similar argument for a two-echelon, *N*-retailer distribution network.

Having constructed an upper bound for the arborescent system, we now construct a single serial system that serves as a lower bound. Here, our approach is similar to Federgruen and Zipkin (1984a) who assume that instantaneous and costless transshipments within an echelon are allowable. The result of such an assumption is that there is only an artificial distinction between installations in an echelon. Each echelon may then be treated as a single virtual installation, where the terminal installation fills all system demands, as shown in Figure 3.

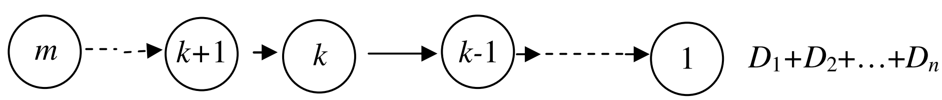

Figure 3: Collapsed Supply Chain

As with the decomposed system, this serial system is solved by the recursive equations (1) and (2). Let $\boldsymbol{s}^{c*}$ and $C^c(\boldsymbol{s}^{c*})$ represent the optimal base-stock policy and expected system wide cost of the collapsed system, respectively. By introducing the same inventory commitment constraints from the decomposed system on echelons 1…*k*-1 of the collapsed system, we achieve the arborescent network. Because the arborescent system is the result of adding constraints on the collapsed network, $C^c(\mathbf{s}^{c*})$ is a lower bound for $C^o(\mathbf{s}^{o*})$.

Additionally, the echelon base stock levels for installations *k*…1 under the collapsed system serve as lower bounds on the sum of echelon base-stock levels for the arborescent network. To see this, consider the echelon immediately downstream of the distribution point (here labeled echelon *k*-1). By combining stages $(k\text{-}1)_1$ and $(k\text{-}1)_2$ from the arborescent network, we gain the opportunity to exploit risk pooling.

Assuming that the chain carries nonnegative safety stocks, the pooling potentially reduces inventory in this installation and thus the optimal echelon base-stock level of the k^{th} echelon.

The decomposition and collapsed system results combine to give

$$C^c\left(s^{c*}\right) \le C^o\left(s^{o*}\right) \le \sum_{\alpha} C_\alpha^d\left(s_\alpha^*\right) \quad \text{and} \quad s_i^{c*} \le s_i^{o*} \le \sum_{\alpha=1}^{n} s_{i,\alpha}^* \quad (\text{for } i = k...m). \tag{6, 7}$$

We use these serial systems to approximate the optimal base-stock levels for the arborescent network. Our approach is to utilize the Shang and Song (2003) heuristic for each of the n+1 constructed chains. Using an illustrative two-retailer system, for the collapsed serial chain system, the stocking level at echelon i is

$$s_i^c = \frac{F_{i,1+2}^{-1}\left(\dfrac{b + \sum_{j=i+1}^{m} h_i}{b + \sum_{j=1}^{m} h_i}\right) + F_{i,1+2}^{-1}\left(\dfrac{b + \sum_{j=i+1}^{m} h_i}{b + \sum_{j=i}^{m} h_i}\right)}{2}, \tag{8}$$

which represents a lower bound for the echelon inventory of the arborescent chain. For the decomposed serial chain system, the stocking levels at echelon i are, for our illustrative system,

$$s_{i,1}^d = \frac{F_{i,1}^{-1}\left(\dfrac{b_1 + \sum_{j=i+1}^{m} h_{i,1}}{b_1 + \sum_{j=1}^{m} h_{i,1}}\right) + F_{i,1}^{-1}\left(\dfrac{b_1 + \sum_{j=i+1}^{m} h_{i,1}}{b_1 + \sum_{j=i}^{m} hi_{i,1}}\right)}{2} \quad \text{and} \quad s_{i,2}^d = \frac{F_{i,2}^{-1}\left(\dfrac{b_2 + \sum_{j=i+1}^{m} h_{i,2}}{b_2 + \sum_{j=1}^{m} h_{i,2}}\right) + F_{i,2}^{-1}\left(\dfrac{b_2 + \sum_{j=i+1}^{m} h_{i,2}}{b_2 + \sum_{j=i}^{m} h_{i,2}}\right)}{2}. \tag{9,10}$$

The sum of these base-stock levels, $s_i^d = s_{i,1}^d + s_{i,2}^d$, represents an upper bound for the echelon inventory of the arborescent chain.

Care must be taken to ensure that if the backorder costs or holding costs differ between installations in chains downstream of echelon k, then this is accounted for when constructing the collapsed system in (8). For installations with generic inventory, our heuristic is to use the weighted average backorder and holding costs. Thus, for a two-retailer system, the combined backordering and holding costs are

$$b = \frac{\mu_1 b_1 + \mu_2 b_2}{\mu_1 + \mu_2} \text{ and } h_i = \frac{\mu_1 h_{i1} + \mu_2 h_{i2}}{\mu_1 + \mu_2} \quad (11, 12)$$

Our heuristic for our stocking level at echelon *i* takes a convex combination of the upper and lower bounds. For simplicity, we use the simple average of the stocking levels,

$$s_i^a = \frac{s_i^c + s_i^d}{2}. \quad (13)$$

5. Simulation Methodology

Because closed-form cost equations do not exist for most common allocation policies, a majority of previous papers on distribution system stocking policies use simulation to test the accuracy of their dynamic programming relaxations. Thus, we also use simulation to test the performance of our approach against prior work and commonly used practitioner heuristics. We consider examples for both symmetric and asymmetric cost structures for two-echelon network topologies with either 2 or 4 terminal retail stages (we briefly discuss in §8.2 related work where we examine the performance for a 3-echelon network).

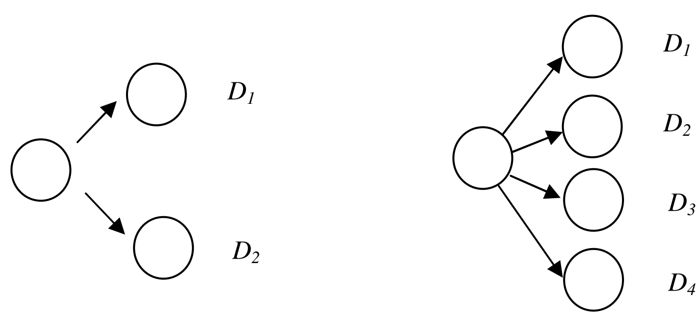

Figure 4: Network Topologies

Our simulation methodology is an unequal variance, two-stage screening-subset selection procedure presented in Nelson et al. (2001). We first create a set of base-stock level candidates covering a range of the expected minimizing base-stock level, +/- at least 5 inventory units for each installation. For the parameter settings in these examples, this range covers approximately 50% of the cumulative distribution of the lead-time demand at each echelon, centered on the cost minimizing stocking level as suggested by the heuristic.

For each set of stocking levels, we conduct a steady-state simulation of 50,000 periods. We batch periods into groups of 10 to reduce deviations from normality and correlations between single-period costs. Based on the lead-times used in our study, we omit the first 10 periods to eliminate initialization effects. The resulting data points are used in the initial screening phase.

Sets of stocking levels that survive the initial screening are subjected to a second round of simulation experiments where we retain our batch mean sizes and generate the number of data points sufficient to eliminate all but one of the systems. After this experiment, the set of stocking levels that has the lowest per period cost is selected. This procedure ensures a confidence level of at least 1-γ that the selected system performs within a quantity g of the optimal system cost. Hence we refer to the selected system as a g–optimal system. For our purposes, we consider $\gamma = 5\%$ and $g = 0.2\%$ of the average per period system cost of the best system found in the first stage.

The simulation model was verified by using the same approach to simulate a serial chain, whereupon the results are identical to those found by Shang and Song (2003). In the next section, we compare the performance of our Bounds Heuristic to other widely used heuristics.

6. Problem Design and Results

6.1. Symmetric Two-Echelon Networks

For the symmetric two-echelon network, we test the heuristics using a partial factorial design over a range of holding cost, backorder cost, and lead-time parameters: $h_{i,\alpha} = \{1,2\}$, $L_i = \{1,2\}$, and $b_\alpha = \{5,10,20\}$. We hold the total periodic system demand, μ, constant at 20 units per period, distributed according to a Poisson distribution. This demand is split among the terminal stages, resulting in $\mu_\alpha = 10$ for the 2-retailer network and $\mu_\alpha = 5$ for the 4-retailer network. These parameters are similar to those used by Jackson (1988), Cachon (2001), Axsater et al. (2002) and Shang and Song (2003), and are summarized in Tables A1 and A2 in Appendix 1 for the two-retailer and four-retailer networks, respectively.

6.1.1 Random Allocation Policies

For the symmetric parameter settings in Tables A1 and A2, we compare the results of the Bounds Heuristic to those of Cachon (2001), whose results are optimal when a random allocation policy is used. These results are presented in Table A3 in Appendix 1 and are summarized in Table 1. Based on this test, we make the following three observations.

% Error Under Random Allocation		
	Two-Retailer	Four-Retailer
Exact	0.00%	0.00%
Bounds	1.41%	1.93%

Table 1: Random Allocation Summary

Observation 1: A small but significant error exists from using the Bounds Heuristic in a random allocation setting. The error grows as the number of retailers increase but the heuristic reacts to parametric changes in a similar manner as the exact analysis.

Observation 2: The exact analysis holds more inventory at the distribution point than the Bounds Heuristic. We discuss possible reasons for this in the next section.

Observation 3: As backorder costs increase, the total system stock held by the Bounds Heuristic falls relative to the exact analysis. For backorder rates of 5, 10, and 20, the exact analysis tends to hold less, equal, and more inventory respectively than the Bounds Heuristic. We discuss this result in the next section.

6.1.2 Myopic Allocation Policies

In this section we compare the systems generated by the Bound Heuristic to the *g*-optimal system found via the simulation procedure described in §5. We also investigate the performance of three alternative heuristics. First, we use the results of Cachon's (2001) exact analysis under random allocation as a heuristic for the myopic allocation problem. Since Graves (1996) finds that holding no safety stock at the upstream stage is frequently a good (and simple) heuristic, we also consider this approach (termed the zero safety stock policy in the results below). Finally, we investigate the performance of setting a fixed service rate at the warehouse stage, as is frequently encountered in practice. We choose a 99% fill rate because, in practice, managers frequently desire high fill rates from the warehouse. The results of these experiments are presented in Tables A4 and A5 in Appendix 1 for the two-retailer and four-retailer networks, respectively. From these results, we make the following observations.

Observation 4: The Bounds Heuristic performs best of all the tested heuristics. It is followed by the exact analysis and zero safety stock heuristics, while the 99% fill-rate heuristic performs poorly in all problems. A summary of these results is presented in Table 2.

% Error Under Myopic Allocation				
	Two-Retailer		Four-Retailer	
Heuristic	Symmetric	Asymmetric	Symmetric	Asymmetric
Newsvendor	0.47%	0.85%	0.66%	0.89%
Cachon	1.75%	NA	2.24%	NA
99% Fill Rate	22.06%	24.59%	21.21%	24.66%
Zero Safety Stock	2.21%	1.96%	2.95%	3.96%

Table 2: Myopic Allocation Summary

Observation 5: The additional upstream inventory held by the exact analysis causes it to under perform the Bounds Heuristic when non-random allocation is allowed. By allocating inventory randomly, the exact analysis increases the variance of the demand placed upon the distribution center by the terminal stages, increasing the required inventory at the second echelon. In contrast, allocating inventory myopically is more efficient. It reduces the penalty induced by preventing the retailers from redistributing inventory, allowing inventory to be placed further downstream, as the efficient allocation from the distribution point will result in less frequent stock outs at the retailers. This effect becomes more important as the backorder cost increase.

Observation 6: All else being held constant, increasing the number of retailers increases the total system cost. Additionally, increasing the holding cost, lead-time, or backorder cost also increases the total system cost. We address these effects further in section 7.

6.2. Asymmetric Two-Echelon Networks

We now consider networks where the terminal stages are asymmetric or non-identical. We consider a partial enumeration over $h_{i,\alpha} = \{1,2\}$, and $b_\alpha = \{5,10,20\}$ for both the 2 and 4-retailer chains, while holding $L_2 = L_1 = 1$ and the system demand as described in section 6.1. The parameters for each problem investigated are presented in Tables A1 and A2. We compare the performance of the Bounds Heuristic, Zero Safety Stock, and 99% Fill Rate heuristics to that of the *g*-optimal system. These results are presented in Tables A6 and A7 in Appendix 1 and are summarized above in Table 2.

Observation 7: Observations 4 and 6 hold in the asymmetric case but asymmetric networks introduce slightly more error in the Bounds Heuristic performance. This increase is present in the other tested heuristics as well, possibly due to a larger number of candidate policies. The Bounds Heuristic returns an average error of 0.87%, while the holding costs between retail locations vary by up to 100% and the backorder costs between locations vary by up to 400%. We believe this range covers most realistic distribution systems.

7. Cost Functions and Analysis of Parameter Effects

As noted by Shang and Song (2003), the simple bounding cost functions presented in §4 enable the analysis of the effects of the system parameters much more readily than previous solution methods. Although these cost functions are general, assuming normally distributed demand allows us to obtain some analytical results. Hence, for Propositions 1 – 3 below, we assume demand at each retailer is normally distributed.

Recall that our method of bounding the arborescent system is through the construction of a set of serial systems. The analysis of the resulting cost functions has a number of parallels to the serial supply chain system studied by Shang and Song (2003). Under symmetric profiles, increasing either the backordering cost or the lead-time increases both total system costs and echelon stocking levels. Increasing the echelon holding cost rate at echelon f increases system costs and stocking levels at echelons below f, while decreasing the echelon base stock levels at and upstream of echelon f. Thus the parametric results for symmetric distribution systems are identical to those of a serial chain.

Proposition 1: for $i = 1, 2, \ldots m$, and where the f^{th} echelon parameter is modified,

(a) as b increases, s_i and $C_i(s_i)$ increase for all i.

(b) as h_f increases, s_i^a increases for i=1,..,f-1 but decreases for i=f, ..,m.

(c) as h_f increases, C_i increases for all i.

(d) as L_f increases, s_i and C_i increase for $i \geq f$.

The analysis becomes slightly more complex when considering asymmetric problems. Here, a change in a given parameter does not affect the echelon stocking levels of an installation that is on a separate path. For the path that does include the modified parameter, the results of Proposition 1 hold. For simplicity, consider an m echelon chain with a single branch point (these results hold for any arborescent chain, although notation becomes burdensome).

Proposition 2. For i= 1, 2, .., m, and α = 1, 2, ... n, with branch point k,

(a) as b_β increases, s_i increases for $i = k, ..., m$; $s_{i,\beta}$ increases for $i = 1, ..., k$-1; and $s_{i,\alpha}$ remains unchanged for $i = 1, \ldots, k$-1 where $\alpha \neq \beta$.

(b) as $h_{f\beta}$ increases, where $f < k$,

a. $s_{i\beta}^l$ decreases while $s_{i\beta}^u$ remains unchanged for $i = f$+1, …, k-1, and s_i^l decreases while s_i^u remains unchanged for $i = k, \ldots, m$.

b. $s_{i\beta}$ increases for $i = 1, \ldots, f$-1 but decreases for $i = f$

c. $s_{i,\alpha}^l$, and $s_{i,\alpha}^u$, remain unchanged for $i = 1, ..., k$-1 and $\alpha \neq \beta$.

(c) as $L_{i,\beta}$ increases, $s_{i,\beta}$ increases for $i = 1, \ldots, k$-1, and s_i increases for $i = k, \ldots, m$ while $s_{i,\alpha}$ remains unchanged for $i = 1, ..., k$-1 and $\alpha \neq \beta$.

Thus increasing the backordering cost at one retailer increases the total system costs and the echelon stocking levels in the sub-chain associated with that retailer; however, the stocking levels of the other sub-chains are independent of the effects of the change in parameter. We find an identical effect for the lead-times between two installations in a sub-chain. For changes in holding cost parameters, we find a similar independence of sub-chains where Proposition 1 describes the resulting changes in system costs and echelon base-stock levels for the affected subchain. The numerical results from our example problems illustrate Propositions 1 and 2 as shown in Figures 5, 6, and 7.

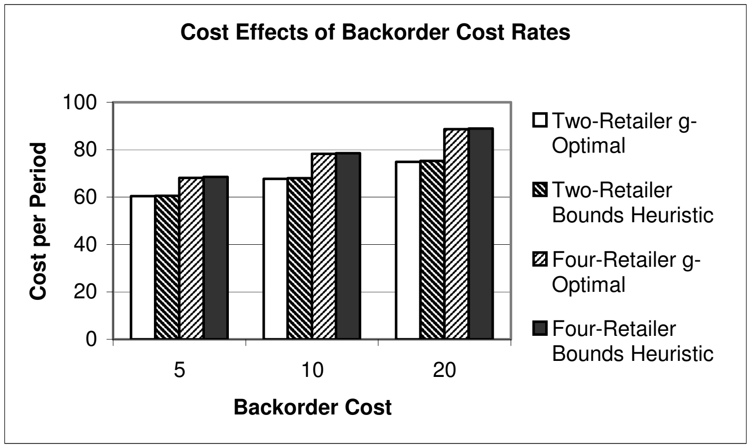

Figure 5: Cost Effects of Backorder Cost Rates

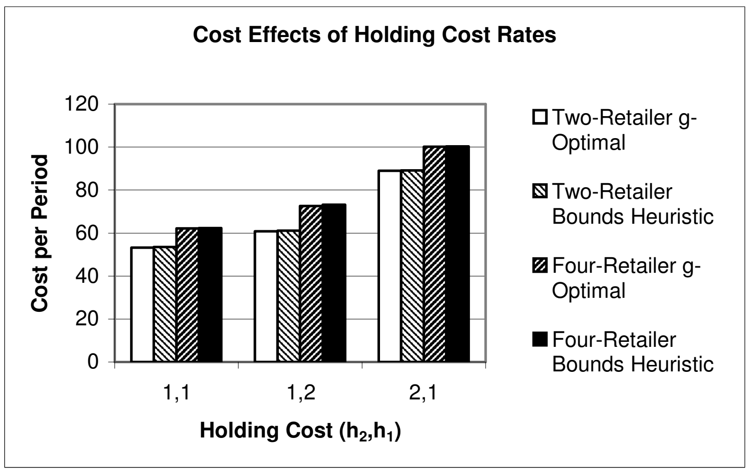

Figure 6: Cost Effects of Holding Cost Rates

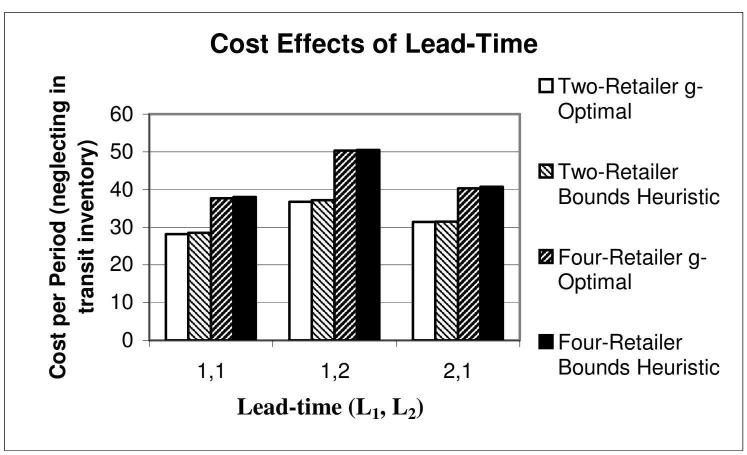

Figure 7: Cost Effects of Lead-Time

Figures 5, 6, and 7 also present the effects of increasing the number of retail locations. Standard risk pooling arguments yield that, keeping the system demand constant, increasing the number of retailers increases both system stocking levels and total system costs. We formalize this intuition in Proposition 3. Consider two distribution systems with $\alpha = 1, \ldots, n$ and $\beta = 1, \ldots, n, n+1$ terminal locations, respectively, and where all other parameters are held constant.

Proposition 3. For $i = 1, 2, ..., m$, $\alpha = 1, 2, ... n$, and $\beta = 1, 2, ... n, n+1$, with branch point k, assuming safety stocks are positive, and keeping all cost and demand parameters constant,

(a) ${s_{i,\alpha}}^c = {s_{i,\beta}}^c$ and ${s_{i,\alpha}}^d > {s_{i,\beta}}^d$ for $i = 1, \ldots, m$.

(b) $\sum_{\alpha=1}^{n} s_{i,\alpha}^{d} < \sum_{\beta=1}^{n+1} s_{i,\beta}^{d}$ for $i = 1, ..., m$

(c) ${C_{i\alpha}}^c = {C_{i\beta}}^c$ for $i = 1, ..., m$

(d) $\sum_{\alpha=1}^{n} C_{i,\alpha}^{d} \leq \sum_{\beta=1}^{n+1} C_{i,\beta}^{d}$ for $i = 1, ..., m$

Proposition 3 states that while increasing the number of retailers in a distribution network (keeping the total system demand constant) reduces the inventory held at each retailer and associated sub-chain, it also increases the total amount of system stock and system cost at each echelon. These effects arise due to the limited ability of the centralized decision maker to exploit risk-pooling opportunities.

Finally, we examine the effects of increasing asymmetry in sub-chain parameters. Because there is no closed form for the inverse of the normal cdf, we condition Propositions 4 - 6 on the assumption of uniform lead-time demand distributions. Our numerical tests verify that the results hold for normal distributions as well, although we note that certain pathological distributions exist for which the results do not hold. We begin by addressing asymmetry in backordering costs. Consider a symmetric chain with a single branch point. Now increase b_1 while decreasing b_2 by an amount, Δ, so that $b_1 = b(1+\Delta)$ and $b_2 = b(1-\Delta)$.

Proposition 4. For i= 1, 2, ..., m, α = 1, 2, ..., n, with branch point k, $b_1 = b(1+\Delta)$ and $b_2 = b(1-\Delta)$, s_i^a decreases as Δ increases and $2{*}C_{i,\alpha} - C_{i,1} - C_{i,2} \geq 0$ and increases with Δ for i = 1, .., m and $\alpha \neq 1,2$.

Proposition 4 states that increasing asymmetry in backordering costs decreases stocking levels and system costs. This seemingly counter intuitive result arises due to the tendency of the system to behave as a serial chain as asymmetry increases. Taken to an extreme, the sub-chain with the high backordering cost captures almost all of the inventory costs. Thus this sub-chain dominates the system as it begins to resemble a serial chain consisting solely of the high backorder cost sub-chain. Recall that the collapsed serial chain serves as a lower bound for the arborescent system. In effect, we find that symmetric sub-chains may be thought of as a 'worst case' scenario for costs. Proposition 4 is illustrated by a small set of test problems, as presented in Figure 8. We compare $2{*}C_{m,\alpha} - C_{m,1} - C_{m,2}$ for two-echelon, two-retailer networks with $b_1 = 15$, $b_2 = 5$ and $b_\beta = 10$ and normally distributed demands with $\mu = 20$ and $\sigma^2 = 20$.

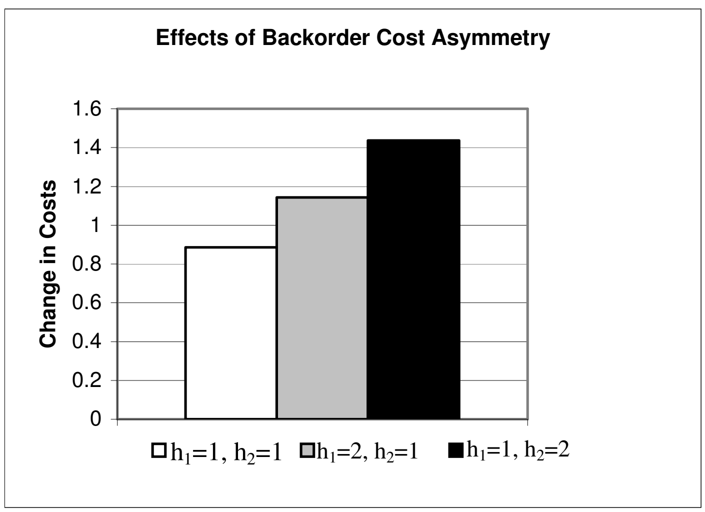

Figure 8: Effects of Backorder Cost Asymmetry

In a similar manner, we examine the effects of asymmetry in the holding costs. We modify the echelon holding cost at echelon f such that $h_{f,1} = h_f(1+\Delta)$ and $h_{f,2} = h_f(1-\Delta)$.

Proposition 5. For i= 1, ..., m, α = 1, ..., n, with branch point k, $h_{f,1} = h_f(1+\Delta)$, and $h_{f,2} = h_f(1-\Delta)$, s_i^a decreases with increasing Δ for i = 1, ..., f-1, and increases for $i = f,\ldots,m$. Also, $C_{i,1} + C_{i,2} \geq 2 * C_{i,\alpha}$ for i = 1, ..., f-1 but $C_{i,1} + C_{i,2} \leq 2 * C_{i,\alpha}$ for i=f, .., m and $\alpha \neq 1,2$.

Proposition 5 states that increasing asymmetry in the holding costs at echelon f decreases (increases) the system echelon stocking levels below (above) echelon f. Additionally, the holding cost asymmetry increases the total system costs when viewed from the perspective of the total chain. When viewed from an echelon below the point of asymmetry, the decrease in stocking levels induced by the asymmetry is associated with increases in the costs applied at the echelon. Hence the presence of asymmetry shifts the application of costs towards the terminal stages, while decreasing the total system costs. Proposition 5 is illustrated by a set of test problems, as presented in Figure 9, where we compare $2^*C_{m,\alpha}$ - $C_{i,1}$ - $C_{i,2}$ for two-echelon, two-retailer networks with $h_{m,1} = 1.5$, $h_{m,2} = 0.5$, $h_{m,\alpha} = 1$ and normally distributed demands with $\mu = 20$ and $\sigma^2 = 20$.

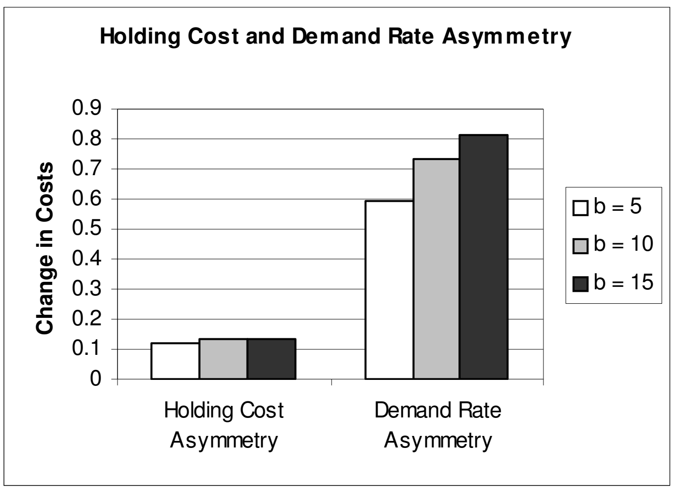

Figure 9: Effects of Holding Cost and Demand Rate Asymmetry

Finally, in Proposition 6 we present the effects of demand asymmetry on the inventory and supply chain costs. The critical fractile computations of our newsvendor approach are independent of demand distribution, hence we return to considering normally distributed demand for this result.

Proposition 6. For $i = 1, ..., m$, $\alpha = 1, ..., n$, branch point k, $\mu_1 = (1+\Delta)\mu_\alpha$, and $\mu_2 = (1-\Delta)\mu_\alpha$, $s_{i,1} + s_{i,2} \leq 2^*s_{i,a}$, and $C_{i,1} + C_{i,2} \leq 2 * C_{i,a}$ for $i = 1, \ldots, m$ and $\alpha \neq 1,2$.

Proposition 6 states that increasing asymmetry in demand rates decreases both echelon stocking levels and system costs. By a similar argument to Proposition 4, the results of Proposition 6 arise due to the tendency of the resulting network to more closely resemble a serial chain. Although the increase in asymmetry decreases the risk pooling savings at the common locations, it also introduces a virtual pooling effect in the lower echelons of the network. A numerical depiction of Proposition 6 is illustrated in Figure 9 where we compare $2*C_{m,\alpha}$ - $C_{m,1}$ - $C_{m,2}$ for two-echelon, two-retailer networks with $\mu_1 = 15$, $\mu_2 = 5$, and $\mu_\alpha = 10$.

8. Extensions

In this section we present three examples of problems from recent papers where the use of our Bounds Heuristic can result in significantly better solutions and insights.

8.1. Nonarborescent Networks

Our technique may be utilized to generate insights beyond the behavior of strictly arborescent network topologies. In this section we present a conceptual routine to determine efficient stocking and allocation policies for directed nonarborescent topologies such as presented below in Figure 10. Problems of this type may be viewed as a traditional distribution system with a relaxation of the arborescence restrictions. As such, their analysis is subject to the difficulties discussed above for distribution systems in addition to their idiosyncratic complexities. Due to these complications, this important class of supply chain problems has been infrequently addressed in the literature. However, the Bounds Heuristic enables tractability and simple analysis of these problems.

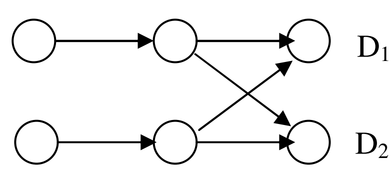

Figure 10: Nonarborescent Distribution System

In a nonarborescent network, each installation may have multiple predecessors and successors, although we retain a hierarchical relationship, ensuring no cyclic shipment possibilities exist in the network. This topology may be interpreted as a distribution network with the possibility of cross-shipping the production, shipment, and sale of substitutable products through multiple retailers, or the utilization of alternate distribution channels for a single product. In this latter interpretation, the installations need not represent physical locations. For instance, consider a local DVD sales chain serving a single market. In this example, two stores serve as retail locations to satisfy customer demand or customers may order the product to be delivered directly to their homes, in which case, either store may fill the order. We depict this network on the left side of Figure 11. Here, D_1 and D_3 represent demand by the customers at the two retail locations while D_2 represents the demand to be shipped to the customers.

Alptekinoglu and Tang (2005) consider a similar distribution network with a series of intermediate crossdocking stages from which each retail installation draws resupply. For tractability, they assume no inventory is held at the intermediate (crossdocking) locations, and that the system supplier is outside of the control of the centralized decision maker. Thus their model resembles that of Eppen and Schrage (1981). Our technique relaxes this assumption, allowing for inventory to be held at all locations, and enabling the associated exploitation of risk pooling and reduced holding cost effects. Their analysis for determining proper stocking and allocation policies relies on a decomposition of the supply chain such that each depot is assigned a certain fraction of the demand at each retailer. This results in a set of arborescent networks where each crossdock serves a set of virtual retail locations. The resulting virtual network is depicted on the right side of Figure 11.

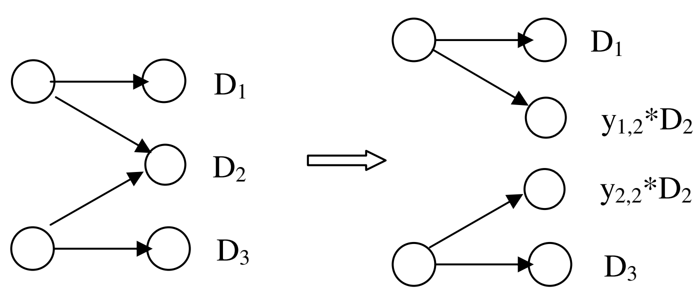

Figure 11: Virtual Distribution Network

Where the decision variable $y_{q,\alpha}$ denotes the fraction installation of installation 1_α's demand that will be assigned to the upper echelon stage q, and where Q upper echelon installations exist. We follow their approach with the Bounds Heuristic. The solution is an optimization problem with a nested approximate cost function based on the heuristic stocking levels. We seek to minimize the sum $\sum_{q=1}^{Q} C_q^*$ subject to $\sum_{q=1}^{Q} y_{q\alpha} = 1$ and nonnegativity constraints, where C_q^* represents the approximately optimal cost of operating the q^{th} network generated by the above decomposition. Although our exposition here is a two-echelon network, the approach may be extended to an arbitrary number of echelons with minimal additional computational (but notationally tedious) complexity.

We note that this cost approximation requires the solution of the nested heuristic base-stock levels as defined in §4. The solution of this optimization problem thus yields the total system stocking levels, but imposes an artificial constraint on the behavior of a supply node. That is, the assignment of a portion $y_{q,\alpha}$ of installation q's demand to installation α is sufficient for a strategic stocking level, but is clearly sub-optimal as an operational allocation policy. Thus, further cost reduction may be achieved through the use of more sophisticated allocation policies.

8.2. Delayed Differentiation

Another context to which the Bounds Heuristic may be applied is in determining the optimal point to delay the differentiation of a product as first explored by Lee and Tang (1997). The reader is referred to a parallel work (Lystad and Ferguson, 2005) where we address this problem more fully. Here we present a brief summary of our findings.

By comparing the behavior of total system costs between three echelon networks with two terminal installations, we determine the reduction in costs achievable by delaying differentiation of a product from the third echelon to the second. We limit our main analysis to inventory costs, holding changes in processing and transportation costs constant. We find that the common use of a decoupling assumption, used by Lee and Tang (1997), frequently overestimates the value of delaying differentiation by overstating inventory pooling effects. This overestimation of value is greatest when echelon holding costs at the point of differentiation are large.

The use of the Bounds Heuristic allows the development of a nonintuitive result in the behavior of the value of delayed differentiation as a function of the local holding costs. In Figure 12, the plotted areas represent the holding cost profile of a unit of product as it progresses through the production process. The graph shows that the value of delaying differentiation from the third echelon to the second is greatest when the holding costs significantly increase at the second echelon. Holding cost increases that occur early in the process, at echelon 3, also indicate significant savings opportunities. Insights such as these cannot be obtained when using the decoupling assumption.

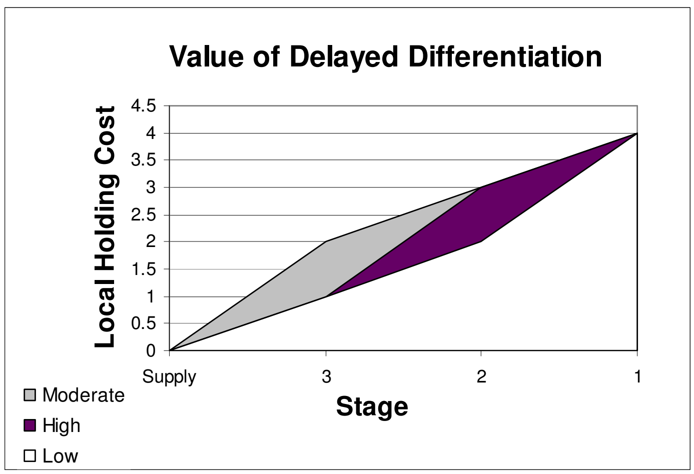

Figure 12: Value of Delayed Differentiation

8.3 Retailer Assignment and Warehouse Location Problems

Another application of the Bounds Heuristic is in the strategic design of a supply chain distribution network. By constructing a distribution network properly (or by redesigning an existing network), companies may reduce logistics, inventory, and transportation costs while simultaneously increasing customer service levels. The objective is to locate warehouses and assign retail installations to each warehouse (or, given an existing network, how to reassign the retail installations to warehouses) such that the total network costs are minimized. The network design and operation problem has been recently addressed in the literature by Chan and Simchi-Levi (1998), Erlebacher and Meller (2000), and Teo and

Shu (2004). A critical limitation of current models is the assumption of a known, deterministic demand rate, where demand must be satisfied without shortage or backordering. Some models include a safety stock parameter to approximate the effects of uncertainty, but this parameter is set exogenously. In practice, the safety stock is typically set to achieve high fill rates, which we have previously shown to be far from optimal in determining the inventory stocking levels. Hence, this practice also leads to distribution network designs that are far from optimal.

Consider a set of $j = 1, \ldots, R$ demand realization points located at coordinates (x_j, y_j) on a 2-dimensional plane. Here points may represent physical retail locations, centroids of regional demands, etc. We assume each of these locations faces stochastic demand with a known distribution. A single supplier exists, located at the origin of the plane. The objective is to identify the number and locations of distribution centers to place on the plane.

For simplicity, assume that transportation costs are linear to the distance traveled and are assessed at an identical rate between locations in a given echelon and the product moves at a constant velocity while in transport, resulting in lead-times between locations that are linearly related to both distance and shipping costs. All products must pass through a distribution center before arriving at a retailer (i.e. no direct shipment option exists from the supplier to a retail point). Also assume no fixed order cost, and linear holding and backordering costs. This, in conjunction with linear transportation costs, results in the optimality of a common base-stock inventory control policy.

To show how the Bounds Heuristic can be incorporated into this problem, we first define the following notation:

t_d, t_r = the linear transportation cost, per unit of product, per unit of distance between the supplier and distribution center, and distribution center and retail location, respectively

n_{xy} = 1 if a distribution center is located at (x,y), and 0 otherwise

a_{jxy} = 1 if retail point j is assigned to a distribution center located at (x,y), and 0 otherwise

F = the fixed cost of operating a distribution center

We formulate a nonlinear optimization problem as

$$\min\left[\sum_{x=-X}^{X}\sum_{y=-Y}^{Y}Fn_{xy}+\sum_{x=-X}^{X}\sum_{y=-Y}^{Y}\left(n_{xy}(x^2+y^2)^{0.5}t_d\sum_{j=1}^{R}\mu_j t_r a_{jxy}((x_i-x)^2+(y_i-y)^2)^{0.5}\right)+C^*(s^*)\right] \quad (14)$$

Subject to the constraints

$$\sum_{x=-X}^{X}\sum_{y=-Y}^{Y}a_{jxy}=1 \text{ for } j = 1, ..., R. \text{ and } \sum_{j=1}^{R}a_{jxy}\le Rn_{xy} \text{ for all } n_{xy}$$

The three terms of (14) are the distribution center annuitized operational costs, total transportation costs, and inventory costs, $C^*(s^*)$. The solution to the inventory cost function, $C^*(s^*)$ may be approximated, given distribution point locations and demand realization assignments, using our Bounds Heuristic described in §4. As discussed in Brandeau and Chiu (1989), warehouse location problems such as (14) suffer from highly nonconvex objectives, inhibiting conventional optimization techniques and the use of dynamic programming techniques to incorporate stochastic demand. The use of the Bounds Heuristic however, allows the inclusion of a sub-routine to determine stocking levels under stochastic demand with just a modest increase in total computing time.

9. Concluding Remarks

In this paper, we present a simple heuristic for arborescent distribution system supply chains with m echelons and n terminal stages. The heuristic requires the computation of $2(n+1)m$ newsvendor problems. Our heuristic performs very well over a wide range of parameters, resulting in an average sub-optimality of less than 0.44% and 0.87% for symmetric and asymmetric cost parameters, respectively, outperforming all other commonly used heuristics. The simplicity of our heuristic also facilitates insights on parametric analysis that are difficult or impossible to obtain using the competing heuristics. For example, we show that the supply chain's inventory and costs increase in the number of retailers, but decrease as backordering costs and demands at the retailers become asymmetric. The results of this work simplify the teaching of supply chain distribution system concepts in the classroom and provide practical insights for managers.

Our heuristic may also be applied in a variety of contexts where a system may be modeled as an arborescent chain. We discuss three such applications, a decomposition of a nonarborescent distribution system, delayed differentiation, and a warehouse location-assignment problem. Our approach enables the analysis of many other historically difficult problems such as the joint inventory-routing problem, the multi-channel supply chain coordination problem, perishable inventory supply chains, and production forecasting under nonstationary demand.

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Appendix 1: Numeric Experimental Data and Results

Two-Echelon Two-Retailer Problem Parameter Settings							
Problem	h_2	$h_{1,1}$	$h_{1,2}$	b_1	b_2	L_2	L_1
1	1	1	1	5	5	1	1
2	1	1	1	10	10	1	1
3	1	1	1	20	20	1	1
4	1	2	2	5	5	1	1
5	1	2	2	10	10	1	1
6	1	2	2	20	20	1	1
7	2	1	1	5	5	1	1
8	2	1	1	10	10	1	1
9	2	1	1	20	20	1	1
10	1	1	1	5	5	1	2
11	1	1	1	10	10	1	2
12	1	1	1	20	20	1	2
13	1	2	2	5	5	1	2
14	1	2	2	10	10	1	2
15	1	2	2	20	20	1	2
16	2	1	1	5	5	1	2
17	2	1	1	10	10	1	2
18	2	1	1	20	20	1	2
19	1	1	1	5	5	2	1
20	1	1	1	10	10	2	1
21	1	1	1	20	20	2	1
22	1	2	2	5	5	2	1
23	1	2	2	10	10	2	1
24	1	2	2	20	20	2	1
25	2	1	1	5	5	2	1
26	2	1	1	10	10	2	1
27	2	1	1	20	20	2	1
28	1	1	1	5	10	1	1
29	1	1	1	5	20	1	1
30	1	1	1	10	20	1	1
31	1	1	2	5	5	1	1
32	1	1	2	10	10	1	1
33	1	1	2	20	20	1	1
34	1	1	2	5	10	1	1
35	1	1	2	5	20	1	1
36	1	1	2	10	20	1	1
37	1	2	1	5	10	1	1
38	1	2	1	5	20	1	1
39	1	2	1	10	20	1	1
40	2	1	1	5	10	1	1
41	2	1	1	5	20	1	1
42	2	1	1	10	20	1	1
43	2	1	2	5	5	1	1
44	2	1	2	10	10	1	1
45	2	1	2	20	20	1	1
46	2	1	2	5	10	1	1
47	2	1	2	5	20	1	1
48	2	1	2	10	20	1	1

Table A1: Two-Retailer Parameter Settings

Two-Echelon, Four-Retailer Network Problem Parameter Settings											
Problem	h_2	$h_{1,1}$	$h_{1,2}$	$h_{1,3}$	$h_{1,4}$	b_1	b_2	b_3	b_4	L_2	L_1
49	1	1	1	1	1	5	5	5	5	1	1
50	1	1	1	1	1	10	10	10	10	1	1
51	1	1	1	1	1	20	20	20	20	1	1
52	1	2	2	2	2	5	5	5	5	1	1
53	1	2	2	2	2	10	10	10	10	1	1
54	1	2	2	2	2	20	20	20	20	1	1
55	2	1	1	1	1	5	5	5	5	1	1
56	2	1	1	1	1	10	10	10	10	1	1
57	2	1	1	1	1	20	20	20	20	1	1
58	1	1	1	1	1	5	5	5	5	1	2
59	1	1	1	1	1	10	10	10	10	1	2
60	1	1	1	1	1	20	20	20	20	1	2
61	1	2	2	2	2	5	5	5	5	1	2
62	1	2	2	2	2	10	10	10	10	1	2
63	1	2	2	2	2	20	20	20	20	1	2
64	2	1	1	1	1	5	5	5	5	1	2
65	2	1	1	1	1	10	10	10	10	1	2
66	2	1	1	1	1	20	20	20	20	1	2
67	1	1	1	1	1	5	5	5	5	2	1
68	1	1	1	1	1	10	10	10	10	2	1
69	1	1	1	1	1	20	20	20	20	2	1
70	1	2	2	2	2	5	5	5	5	2	1
71	1	2	2	2	2	10	10	10	10	2	1
72	1	2	2	2	2	20	20	20	20	2	1
73	2	1	1	1	1	5	5	5	5	2	1
74	2	1	1	1	1	10	10	10	10	2	1
75	2	1	1	1	1	20	20	20	20	2	1
76	1	1	1	1	1	5	5	10	10	1	1
77	1	1	1	1	1	5	5	20	20	1	1
78	1	1	1	1	1	10	10	20	20	1	1
79	1	1	2	1	2	5	5	5	5	1	1
80	1	1	2	1	2	10	10	10	10	1	1
81	1	1	2	1	2	20	20	20	20	1	1
82	1	1	2	1	2	5	5	10	10	1	1
83	1	1	2	1	2	5	5	20	20	1	1
84	1	1	2	1	2	10	10	20	20	1	1
85	2	1	1	1	1	5	5	10	10	1	1
86	2	1	1	1	1	5	5	20	20	1	1
87	2	1	1	1	1	10	10	20	20	1	1
88	2	1	2	1	2	5	5	5	5	1	1
89	2	1	2	1	2	10	10	10	10	1	1
90	2	1	2	1	2	20	20	20	20	1	1
91	2	1	2	1	2	5	5	10	10	1	1
92	2	1	2	1	2	5	5	20	20	1	1
93	2	1	2	1	2	10	10	20	20	1	1

Table A2: Four-Retailer Parameter Settings

Random Allocation Policy Results							
Problem	Exact Results			Bounds Heuristic Results			
	s_2	s_1	Cost	s_2	s_1	Cost	% Error
1	41	22	48.15	40	23	48.26	0.22
2	43	24	55.29	39	25	56.44	2.08
3	44	26	62.28	40	26	64.57	3.68
4	42	21	54.22	41	22	54.76	0.99
5	43	23	64.32	42	23	64.47	0.23
6	44	25	74.58	43	24	75.52	1.26
7	38	22	77.35	35	24	79.12	2.29
8	41	23	88.56	37	25	90.19	1.84
9	42	25	99.71	38	26	102.27	2.56
10	40	33	72.42	37	35	73.34	1.27
11	42	35	80.75	39	36	81.34	0.73
12	43	37	88.89	38	38	91.70	3.16
13	41	31	79.95	40	33	80.87	1.15
14	42	34	91.80	40	35	92.72	1.01
15	44	36	103.67	41	36	104.78	1.07
16	38	32	122.56	34	35	124.40	1.50
17	40	34	135.45	34	37	137.96	1.85
18	42	36	148.42	35	38	152.71	2.89
19	62	22	49.91	61	23	49.91	0.01
20	64	24	57.38	61	25	57.88	0.87
21	66	26	64.69	62	26	66.15	2.25
22	62	21	56.03	62	22	56.48	0.80
23	64	23	66.50	64	23	66.50	0.00
24	66	25	77.04	65	24	77.84	1.04
25	58	22	79.85	56	24	81.08	1.54
26	60	24	91.86	58	25	92.62	0.82
27	63	25	103.76	60	26	104.69	0.89
49	41	11	57.35	40	12	58.11	1.33
50	43	12	67.60	39	13	68.46	1.27
51	44	14	77.35	39	14	80.19	3.67
52	42	10	65.74	42	11	66.90	1.76
53	43	12	80.28	42	12	80.31	0.04
54	45	13	94.32	43	13	94.68	0.38
55	38	11	89.05	33	13	93.67	5.19
56	40	12	104.10	36	13	104.19	0.09
57	42	13	119.62	36	14	124.19	3.82
58	39	17	83.73	37	18	84.94	1.44
59	42	18	95.32	34	20	98.56	3.39
60	43	20	106.86	34	21	112.41	5.20
61	40	16	94.19	39	17	96.40	2.35
62	43	17	110.60	40	18	111.80	1.09
63	44	19	127.73	41	19	128.53	0.63
64	37	16	136.55	29	19	140.77	3.09
65	40	17	154.55	30	20	158.54	2.58
66	41	19	172.19	30	21	179.29	4.12
67	62	11	58.94	61	12	59.52	0.99
68	63	13	69.47	62	13	69.47	0.01
69	65	14	79.40	62	14	80.25	1.07
70	63	10	67.45	64	11	68.78	1.93
71	64	12	82.15	64	12	82.15	0.00
72	66	13	96.55	65	13	96.69	0.14
73	57	11	91.15	53	13	94.71	3.90
74	61	12	106.99	59	13	108.29	1.22
75	63	13	123.20	58	14	125.13	1.56

Table A3: Random Allocation Results

Myopic Allocation Policy Results for 2-Echelon, 2-Retailer Symmetric Networks															
Problem	g-Optimal System			Bounds Heuristic			Cachon Heurisitc			99% Fill Rate Heuristic			Zero Safety Stock Heuristic		
	s_2	s_1	Cost	s_2	s_1	%Error	s_2	s_1	%Error	s_2	s_1	%Error	s_2	s_1	%Error
1	17	14	38.53	20	13	0.50	21	12	1.21	31	13	23.44	20	13	0.50
2	20	14	43.47	19	15	0.54	23	14	2.49	31	15	23.12	20	15	1.32
3	18	16	47.82	20	16	1.63	24	16	6.33	31	16	20.14	20	16	1.63
4	19	12	42.88	21	12	0.96	22	11	1.48	31	12	20.23	20	12	0.31
5	21	13	49.99	22	13	0.62	23	13	1.42	31	13	14.93	20	13	0.45
6	22	14	57.40	23	14	0.28	24	15	3.35	31	14	10.95	20	14	1.69
7	14	14	63.66	15	14	0.18	18	12	0.65	31	14	39.90	20	14	7.61
8	16	15	71.22	17	15	0.12	21	13	1.73	31	15	33.03	20	15	4.66
9	19	15	78.75	18	16	0.64	22	15	2.90	31	16	27.71	20	16	2.73
10	14	26	64.11	17	25	0.29	20	23	0.79	31	25	18.09	20	25	2.49
11	18	26	70.70	19	26	0.23	22	25	1.19	31	26	13.63	20	26	0.60
12	18	28	76.18	18	28	1.27	23	27	2.79	31	28	14.15	20	28	1.83
13	17	24	70.60	20	23	0.46	21	21	1.06	31	23	13.77	20	23	0.46
14	17	26	80.13	20	25	0.24	22	24	0.30	31	25	11.97	20	25	0.24
15	19	27	88.09	21	26	1.63	24	26	3.22	31	26	10.35	20	26	1.59
16	11	26	110.56	14	25	0.22	18	22	0.93	31	25	24.27	20	25	5.62
17	9	29	120.20	14	27	0.28	20	24	1.23	31	27	22.93	20	27	5.50
18	15	28	130.40	15	28	0.00	22	26	1.82	31	28	18.22	20	28	2.71
19	40	13	40.64	41	13	0.27	42	12	1.32	55	13	26.97	40	13	0.23
20	40	15	46.05	41	15	0.51	44	14	1.09	55	15	24.97	40	15	0.34
21	41	16	51.35	42	16	-0.08	46	16	3.63	55	16	19.73	40	16	0.84
22	40	12	45.20	42	12	0.35	42	11	1.13	55	12	22.99	40	12	0.23
23	43	13	52.72	44	13	0.38	44	13	0.38	55	13	16.65	40	13	2.09
24	45	14	60.46	45	14	0.00	46	15	2.24	55	14	12.03	40	14	6.02
25	35	14	66.91	36	14	0.32	38	12	0.99	55	14	45.13	40	14	5.07
26	37	15	75.32	38	15	0.22	40	14	0.34	55	15	36.48	40	15	2.19
27	39	16	83.70	40	16	0.69	43	15	1.38	55	16	29.78	40	16	0.69

Table A4: Myopic Allocation Policy Results for Two-Retailer Symmetric Networks

Myopic Allocation Policy Results for 2-Echelon, 4-Retailers Symmetric Networks															
Problem	g-Optimal System			Bounds Heuristic			Cachon Heuristic			99% Fill Rate Heuristic			Zero Safety Stock Heuristic		
	s_2	s_1	Cost	s_2	s_1	%Error	s_2	s_1	% Error	s_2	s_1	%Error	s_2	s_1	%Error
49	17	7	44.4	20	7	1.34	21	6	2.6	31	7	22	20	7	1.336
50	18	8	51.3	19	8	0.21	23	7	3.06	31	8	18.7	20	8	0.814
51	18	9	58.3	19	9	0.1	24	9	5.02	31	9	16.4	20	9	0.622
52	19	6	50.5	22	6	1.55	22	5	4.79	31	6	16.6	20	6	0.241
53	20	7	60.4	22	7	0.87	23	7	1.71	31	7	13.2	20	7	0
54	20	8	70.7	23	8	1	25	8	2.93	31	8	10.8	20	8	0
55	15	7	70.8	13	8	0.66	18	6	1.14	31	8	44	20	8	13.87
56	15	8	81	16	8	0.16	20	7	1.45	31	8	28.8	20	8	4.331

57	16	9	91.6	16	9	0	22	8	2.49	31	9	25.5	20	9	3.707
58	15	13	72.7	17	13	0.26	19	12	0.59	31	13	15.7	20	13	2.136
59	17	14	81.7	14	15	0.27	22	13	1.71	31	15	17.5	20	15	4.398
60	14	16	90.8	14	16	0	23	15	2.14	31	16	14.3	20	16	2.836
61	16	12	81.1	20	12	1.51	20	11	0.7	31	12	13.5	20	12	1.505
62	18	13	94.4	20	13	0.3	23	12	1.78	31	13	10.1	20	13	0.297
63	19	14	108	21	14	0.1	24	14	1.57	31	14	7.73	20	14	0
64	0	16	121	9	14	0.08	17	11	1.29	31	14	28.7	20	14	10.96
65	10	15	135	10	15	0	20	12	2.2	31	15	24.3	20	15	8.429
66	3	18	149	10	16	0.05	21	14	1.32	31	16	20.5	20	16	6.224
67	38	7	46.2	41	7	1.17	42	6	3.08	55	7	26	40	7	0.64
68	39	8	53.5	42	8	1.60	43	8	2.05	55	8	21.5	40	8	0.107
69	40	9	60.7	42	9	0.54	45	9	3.35	55	9	18.3	40	9	0
70	40	6	52.5	44	6	1.81	43	5	5.07	55	6	19.6	40	6	0
71	41	7	62.8	44	7	1	44	7	1	55	7	15.4	40	7	0.546
72	42	8	73.3	45	8	0.9	46	8	1.68	55	8	12.3	40	8	1.428
73	31	8	73.5	33	8	0.64	37	6	1.97	55	8	49.5	40	8	11.03
74	36	8	84.5	39	8	0.93	41	7	1.7	55	8	33.1	40	8	2.483
75	37	9	95.6	38	9	0.14	43	8	2.16	55	9	28.6	40	9	1.783

Table A5: Myopic Allocation Policy Results for Four-Retailer Symmetric Networks

Myopic Allocation Policy Results for Two-Echelon, Two-Retailer Asymmetric Networks																
Problem	g-Optimal System				Bound Heuristic				99% Fill Rate Heuristic				Zero Safety Stock Heuristic			
	s_2	$s_{1,1}$	$s_{1,2}$	Cost	s_2	$s_{1,1}$	$s_{1,2}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	% Error
28	19	13	14	40.97	20	13	15	1.31	31	13	15	23.33	20	13	15	1.31
29	19	13	16	43.42	21	13	16	1.26	31	13	16	20.85	20	13	16	0.45
30	19	15	16	45.93	20	15	16	0.92	31	15	16	20.75	20	15	16	0.64
31	19	13	12	40.66	20	13	12	0.39	31	13	12	21.84	20	13	12	0.37
32	20	14	13	46.73	21	15	13	0.91	31	15	13	18.77	20	15	13	0.47
33	22	15	14	52.88	22	16	14	0.72	31	16	14	14.33	20	16	14	0.79
34	20	13	13	44.29	21	13	13	0.46	31	13	13	18.55	20	13	13	0.00
35	19	13	15	47.97	22	13	14	1.02	31	13	14	16.19	20	13	14	0.50
36	21	14	14	50.41	21	15	14	0.65	31	15	14	16.34	20	15	14	1.15
37	20	12	14	43.18	20	12	15	1.19	31	12	15	21.61	20	12	15	0.80
38	19	12	16	45.65	21	12	16	0.97	31	12	16	19.51	20	12	16	0.53
39	21	13	15	49.21	21	13	16	0.61	31	13	16	16.60	20	13	16	0.69
40	15	14	15	67.39	16	14	15	0.27	31	14	15	36.33	20	14	15	5.87
41	17	13	16	71.27	18	14	16	1.51	31	14	16	32.96	20	14	16	4.92
42	16	15	16	74.98	17	15	16	0.35	31	15	16	30.21	20	15	16	3.26
43	17	13	12	65.47	17	14	12	0.59	31	14	12	34.76	20	14	12	4.62
44	18	14	13	74.04	18	15	13	0.41	31	15	13	28.18	20	15	13	2.17
45	18	16	15	82.93	19	16	15	0.79	31	16	15	25.21	20	16	15	1.79
46	16	13	14	70.35	18	14	13	1.15	31	14	13	30.79	20	14	13	3.33
47	17	13	15	75.33	18	14	15	1.33	31	14	15	30.06	20	14	15	3.47
48	18	14	15	79.09	18	15	15	0.52	31	15	15	27.57	20	15	15	2.49

Table A6: Myopic Allocation Policy Results for Two-Retailer Asymmetric Networks

Myopic Allocation Policy Results for Two-Echelon, Four-Retailer Asymmetric Networks																								
Problem	g-Optimal System						Bounds Heuristic						99% Fill Rate Heuristic						Zero Safety Stock Heuristic					
	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	Cost	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	% Error
76	18	7	7	8	8	47.8	19	7	7	8	8	0.5	31	7	7	8	8	20.4	20	7	7	8	8	1.2
77	18	7	7	9	9	51.2	20	7	7	9	9	1.0	31	7	7	9	9	19.1	20	7	7	9	9	1.0
78	18	8	8	9	9	54.7	19	8	8	9	9	0.3	31	8	8	9	9	17.6	20	8	8	9	9	0.8
79	19	7	6	7	6	47.4	21	7	6	7	6	1.3	31	7	6	7	6	19.1	20	7	6	7	6	0.5
80	19	8	7	8	7	55.8	20	8	7	8	7	0.4	31	8	7	8	7	15.9	20	8	7	8	7	0.4
81	19	9	8	9	8	64.5	21	9	8	9	8	0.7	31	9	8	9	8	13.3	20	9	8	9	8	0.3
82	18	7	6	8	7	51.6	21	7	6	8	7	1.2	31	7	6	8	7	17.4	20	7	6	8	7	0.5
83	19	7	6	9	8	55.8	21	7	6	9	8	1.2	31	7	6	9	8	16.1	20	7	6	9	8	0.2
84	19	8	7	9	8	60.1	21	8	7	9	8	1.0	31	8	7	9	8	14.6	20	8	7	9	8	0.4
85	15	7	7	8	8	75.9	14	8	8	8	8	0.4	31	8	8	8	8	36.0	20	8	8	8	8	8.5
86	15	7	7	9	9	81.0	16	8	8	9	9	2.0	31	8	8	9	9	33.9	20	8	8	9	9	8.2
87	15	8	8	9	9	86.2	16	8	8	9	9	0.3	31	8	8	9	9	27.4	20	8	8	9	9	4.2
88	15	7	7	7	6	73.2	14	8	7	8	7	0.9	31	8	7	8	7	40.2	20	8	7	8	7	11.2
89	15	8	8	8	7	84.9	16	8	8	8	8	1.0	31	8	8	8	8	30.3	20	8	8	8	8	6.2
90	16	9	9	9	8	97.1	16	9	9	9	9	0.9	31	9	9	9	9	26.8	20	9	9	9	9	5.5
91	16	7	6	8	7	79.0	15	8	7	8	8	1.2	31	8	7	8	8	35.0	20	8	7	8	8	8.5
92	16	7	6	9	8	85.1	15	8	7	9	9	0.9	31	8	7	9	9	32.6	20	8	7	9	9	7.8
93	17	8	7	9	8	91.0	16	8	8	9	9	0.9	31	8	8	9	9	28.4	20	8	8	9	9	5.8

Table A7: Myopic Allocation Policy Results for Four-Retailer Asymmetric Networks

Appendix 2: Proofs of Propositions 1-6

Define the critical fractiles

$$\Theta_{i,\alpha}^{l,d} = \frac{b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}}{b_\alpha + \sum_{j=1}^{m} h_{j,\alpha}} \quad \Theta_{i,\alpha}^{u,d} = \frac{b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}}{b_\alpha + \sum_{j=i}^{m} h_{j,\alpha}} \quad \Theta_{i}^{l,c} = \frac{b + \sum_{j=i+1}^{m} h_{j}}{b + \sum_{j=1}^{m} h_{j}} \quad \Theta_{j}^{u,c} = \frac{b + \sum_{j=i+1}^{m} h_{j}}{b + \sum_{j=i}^{m} h_{j}}$$

Let

$z_{i,\alpha}^{l,d} = \Phi^{-1}\left(\Theta_{i,\alpha}^{l,d}\right)$, $z_{i,\alpha}^{u,d} = \Phi^{-1}\left(\Theta_{i,\alpha}^{u,d}\right)$, $z_{i}^{l,c} = \Phi^{-1}\left(\Theta_{i}^{l,c}\right)$, and $z_{i}^{u,c} = \Phi^{-1}\left(\Theta_{i}^{u,c}\right)$.

Let $\phi(\cdot)$ and $\Phi(\cdot)$ represent the standard normal pdf and cdf, respectively. Following the approach in Zipkin (2000) (see also Shang and Song, 2003),

$$s_{i,\alpha}^{l,d} = \mu_\alpha \tilde{L}_{i,\alpha} + z_{i,\alpha}^{l,d}\sqrt{\sigma_\alpha{}^2 \tilde{L}_{i,\alpha}} \tag{A1}$$

$$s_{i}^{l,d} = \sum_{\alpha=1}^{n} s_{i,\alpha}^{l,d} \tag{A2}$$

$$s_{i,\alpha}^{u,d} = \mu_\alpha \tilde{L}_{i,\alpha} + z_{i,\alpha}^{u,d}\sqrt{\sigma_\alpha{}^2 \tilde{L}_{i,\alpha}} \tag{A3}$$

$$s_{i}^{u,d} = \sum_{\alpha=1}^{n} s_{i,\alpha}^{u,d} \tag{A4}$$

$$s_{i}^{l,c} = \mu\ \tilde{L}_i + z_{i}^{l,c}\sqrt{\sigma^2 \tilde{L}_i} \tag{A5}$$

$$s_{i}^{u,c} = \mu\ \tilde{L}_i + z_{i}^{u,c}\sqrt{\sigma^2 \tilde{L}_i} \tag{A6}$$

$$s_{i}^{a} = \mu\ \tilde{L}_i + \frac{(z_i^{u,c} + z_i^{l,c})}{4}\sqrt{\sigma^2 \tilde{L}_i} + \sum_{\alpha=1}^{n}\frac{(z_{i,\alpha}^{u,d} + z_{i,\alpha}^{l,d})}{4}\sqrt{\sigma_\alpha{}^2 \tilde{L}_{i,\alpha}} \tag{A7}$$

$$C_{i,\alpha}^{l,d}\left(s_{i,\alpha}^{u,d}\right) = \left(b_\alpha + \sum_{j=i}^{m} h_{i,\alpha}\right)\phi\left(z_{i,\alpha}^{u,d}\right)\sqrt{\sigma_\alpha{}^2 \tilde{L}_{i,\alpha}} + \sum_{j=1}^{i-1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_{i,\alpha} \tag{A8}$$

$$C_{i}^{l,d} = \sum_{\alpha=1}^{n} C_{i,\alpha}^{l,d} \tag{A9}$$

$$C_{i,\alpha}^{u,d}\left(s_{i,\alpha}^{l,d}\right) = \left(b_\alpha + \sum_{j=1}^{m} h_{i,\alpha}\right)\phi\left(z_{i,\alpha}^{l,d}\right)\sqrt{\sigma_\alpha{}^2 \tilde{L}_{i,\alpha}} + \sum_{j=1}^{i-1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_{i,\alpha} \tag{A10}$$

$$C_{i}^{u,d} = \sum_{\alpha=1}^{n} C_{i,\alpha}^{u,d} \tag{A11}$$

$$C_{i}^{l,c}\left(s_{i}^{u,c}\right) = \left(b + \sum_{j=i}^{m} h_{i}\right)\phi\left(z_{i}^{u,c}\right)\sqrt{\sigma^2 \tilde{L}_i} + \sum_{j=1}^{i-1} h_{j+1}\mu \tilde{L}_i \tag{A12}$$

$$C_i^{u,c}\left(s_i^{l,c}\right)=\left(b+\sum_{j=1}^{m}h_i\right)\phi\left(z_i^{l,c}\right)\sqrt{\sigma^2\tilde{L}_i}+\sum_{j=1}^{i-1}h_{j+1}\mu\tilde{L}_i \qquad \text{(A13)}$$

In the following proofs, we occasionally will suppress notation when the context is clear. For example, we use the notation s_i^l to represent the stocking level of the lower bound functions A1, A2, and A5 (that is, $s_{i,\alpha}^{l,d}$, $s_i^{l,d}$, and $s_i^{l,c}$) when the proof applies to each or when each variable behaves similarly.

Proof of Proposition 1.

Propositions 1a, 1b, 1c, and 1d may be determined by inspection of equations (A1) through (A13). Consider

(a) Equations A1 through A7 are increasing in *z*, which is increasing in *b*. Additionally, the cost equations A8, A10, A12, and A13 increase in both z and *b*.

(b) We investigate five cases, s_i^l for $1 \le i < f$ and $i \ge f$, and s_i^u for $1 \le i < f$, $i = f$, and $i > f$.

(b1) For $1 \le i < f$, the numerator and denominator of critical fractiles $\Theta_{i,\alpha}^{l,d}$ and $\Theta_{i,\alpha}^{l,c}$ both increase by the change in h_f. Hence equations A1, A2, and A5 increase.

(b2) For $i \ge f$, the denominator of critical fractiles $\Theta_{i,\alpha}^{l,d}$ and $\Theta_{i,\alpha}^{l,c}$ increases by the change in h_f. Hence equations A1, A2, and A5 decrease.

(b3) For $1 \le i < f$, the numerator and denominator of critical fractiles $\Theta_{i,\alpha}^{u,d}$ and $\Theta_{i,\alpha}^{u,c}$ both increase by the change in h_f. Hence equations A3, A4, and A6 increase.

(b4) For $i = f$, the denominator of critical fractiles $\Theta_{i,\alpha}^{u,d}$ and $\Theta_{i,\alpha}^{u,c}$ increases by the change in h_f. Hence equations A3, A4, and A6 decrease.

(b5) For $i > f$, critical fractiles $\Theta_{i,\alpha}^{u,d}$ and $\Theta_{i,\alpha}^{u,c}$ are independent of h_f and are thus unaffected by a change. Hence equations A3, A4 and A6 are unaffected.

Combining (b1) through (b5) yields the result.

(c) Examination of equations A8, A10, A12, and A13 shows that for a fixed y, $C_i(y)$ increases with h_f for all i due to the increase in either the first coefficient or last term for A8 and A12, and at least one of these for A10 and A13.

(d) Equations A1 through A7 and A8 through A13 are increasing in $\tilde{L}_i$, which is increasing in L_f for $i \geq f$.

Proof of Proposition 2

Proposition 2 follows by inspection of equations A1 through A11 and the observations in Proposition 1.

(a) We consider three cases, $i < k$, and $i \geq k$ for the decomposed system and all i for the collapsed system.

(a1) For $i < k$ in the decomposed system we note that the subchains are independent. Hence changes in b_β do not affect Equations A1 through A4 and A7 through A11 for $\alpha \neq \beta$..

(a2) For $i \geq k$ in the decomposed system, note that the stocking equations A1 and A3 and the cost equations A8 and A10 are independent of changes in b_β for $\alpha \neq \beta$.. For $\alpha = \beta$, Proposition1A holds. Hence equations A2, A4, A9, and A11 increase with b_β.

(a3) For the collapsed system, note that increasing b_β increases $b = \dfrac{1}{\mu}\sum_{\alpha=1}^{n}(\mu_\alpha b_\alpha)$.

Hence z and b increase, increasing equations A5, A6, A12, and A13.

(b) We consider three cases, $i < f$, $f \leq i < k$, and $i \geq k$

(b1) For $i < f$, we consider two cases, $\alpha \neq \beta$ and $\alpha = \beta$.

(b1a) For $\alpha \neq \beta$, stocking equations A1 and A3 and cost equations A8 and through A10 are independent of $h_{f,\beta}$. Increasing $h_{f,\beta}$ increases equations A5 and A6, as the critical fractiles increase. Additionally, equations A12 and A13 increase.

(b1b) For $\alpha = \beta$, Propositions 1b1 and 1b3 hold.

Hence as $h_{f,\beta}$ increases, for $i < f$, stocking equations A2, A4, A5, and A6 increase, and cost equations A9, A11, A12, and A13 increase.

(b2) For $f \leq i < k$ we consider two cases, $\alpha \neq \beta$ and $\alpha = \beta$.

(b2a) For $\alpha \neq \beta$, stocking equations A1 and A3 and cost equations A8 and through A10 are independent of $h_{f,\beta}$. Increasing $h_{f,\beta}$ decreases $z_i^{l,c}$ hence equation

A5 decreases. Increasing $h_{f,\beta}$ either decreases $z_i^{u,c}$ or leaves $z_i^{u,c}$ unchanged, if $i=f$ or $i>f$, respectively. Hence equation A6 either decreases or remains independent of changes in $h_{f,\beta}$. As noted in proposition 1c, these effects cause equations A12 and A13 to increase.

(b1b) For $\alpha=\beta$, Propositions 1b2, 1b4, and 1b5 hold.

Hence stocking equations A2, A4, A5, and A6 decrease while cost equations A9, A11, A12, and A13 increase.

(b3) For $i \geq k$ we consider two cases, the decomposed and collapsed systems.

(b3a) For the decomposed systems, Proposition 2b2 holds for $i \geq k$.

(b3b) For the collapsed system, increasing $h_{f,\beta}$ increases $h_f = \frac{1}{\mu}\sum_{\alpha=1}^{n}\left(\mu_\alpha h_{f,i}\right)$ and hence equations A5 and A6 decrease while equations A12 and A13 increase.

(c) We consider two cases, $\alpha \neq \beta$ and $\alpha=\beta$.

(c1) for $\alpha \neq \beta$, equations A1, A3, A8, and A10 are independent of $L_{i,\beta}$ for all i. Equations A5, A6, A12, and A13 all increase with $L_{i,\beta}$ for all i.

(c2) for $\alpha = \beta$ equations A1, A3, A8, and A10 increase with $L_{i,\beta}$ for all i. Equations A5, A6, A12, and A13 all increase with $L_{i,\beta}$ for all i.

Hence an increase in $L_{i,\beta}$ increases equations A1 through A13.

Proof of Proposition 3.

In this proposition, we hold the total system demand process constant. Assuming demand is distributed normally with mean μ and variance σ^2, splitting among n identical terminal locations gives $\mu = \sum_{\alpha=1}^{n}\mu_\alpha$ and $\sigma^2 = \sum_{\alpha=1}^{n}\sigma_\alpha^2$, or $\mu_\alpha = \frac{\mu}{n}$ and $\sigma_\alpha^2 = \frac{\sigma^2}{n}$ while splitting the demand process across n+1 identical terminal locations gives $\mu_\beta = \frac{\mu}{n+1}$ and $\sigma_\beta^2 = \frac{\sigma^2}{n+1}$.

(a) We consider two cases, the decomposed and collapsed systems.

(a1) For the decomposed system, $\mu_\beta < \mu_\alpha$ and $\sigma_\beta^2 < \sigma_\alpha^2$, hence $s_{i,\alpha}^d > s_{i,\beta}^d$ from equations A1 and A3.

(a2) For the collapsed system, $\mu = \sum_{\alpha=1}^{n}\mu_\alpha$ and $\sigma^2 = \sum_{\alpha=1}^{n}\sigma_\alpha^2$ remain unchanged. Hence Equations A5 and A6 remain unchanged.

(b) Consider $\sum_{\alpha=1}^{n} s_{i,a}^{d} - \sum_{\beta=1}^{n+1} s_{i,\beta}^{d}$. From equations A1 or A3,

$$= \sum_{\alpha=1}^{n}\left(\mu_{\alpha}\tilde{L}_i + z_i^d \sqrt{\sigma_{\alpha}{}^2 \tilde{L}_i}\right) - \sum_{\beta=1}^{n+1}\left(\mu_{\beta}\tilde{L}_i + z_i^d \sqrt{\sigma_{\beta}{}^2 \tilde{L}_i}\right)$$

$$= \sum_{\alpha=1}^{n}\left(\mu_{\alpha}\tilde{L}_i + z_i^d \sqrt{\sigma_{\alpha}{}^2 \tilde{L}_i}\right) - \sum_{\beta=1}^{n+1}\left(\frac{n\mu_{\alpha}}{n+1}\tilde{L}_i + z_i^d \sqrt{\frac{n\sigma_{\alpha}{}^2}{n+1}\tilde{L}_i}\right)$$

$$= \sum_{\alpha=1}^{n}\left(z_i^d \sqrt{\sigma_{\alpha}{}^2 \tilde{L}_i}\right) - \sum_{\beta=1}^{n+1}\left(z_i^d \sqrt{\frac{n\sigma_{\alpha}{}^2}{n+1}\tilde{L}_i}\right)$$

$$= n\left(z_i^d \sqrt{\sigma_{\alpha}{}^2 \tilde{L}_i}\right) - (n+1)\left(z_i^d \sqrt{\frac{n\sigma_{\alpha}{}^2}{n+1}\tilde{L}_i}\right)$$

$$= z_i^d \sqrt{\tilde{L}_i}\left(n\sqrt{\sigma_{\alpha}^2} - (n+1)\sqrt{\frac{n\sigma_{\alpha}^2}{n+1}}\right)$$

$$= z_i^d \sqrt{\tilde{L}_i}\left(\sqrt{n^2\sigma_{\alpha}^2} - \sqrt{(n+1)n\sigma_{\alpha}^2}\right)$$

$$< 0$$

(c) $\mu = \sum_{\alpha=1}^{n} \mu_{\alpha}$ and $\sigma^2 = \sum_{\alpha=1}^{n} \sigma_{\alpha}^2$ remain unchanged. Hence Equations A12 and A13 remain unchanged.

(d) This proof follows similar algebra as Proposition 3b. Let K be a positive constant equal to $\left(b + \sum_{j=i}^{m} h_i\right)\phi\left(z_i^{u,c}\right)$. Then

$$\sum_{\alpha=1}^{n} C_{i,a}^{l,d} - \sum_{\beta=1}^{n+1} C_{i,\beta}^{l,d}$$

$$= \sum_{\alpha=1}^{n}\left(K\sqrt{\sigma_{\alpha}{}^2\tilde{L}_i} + \sum_{j=1}^{i=1} h_{j+1}\mu_{\alpha}\tilde{L}_i\right) - \sum_{\beta=1}^{n+1}\left(K\sqrt{\sigma_{\beta}{}^2\tilde{L}_i} + \sum_{j=1}^{i=1} h_{j+1}\mu_{\beta}\tilde{L}_i\right)$$

$$= \sum_{\alpha=1}^{n}\left(K\sqrt{\sigma_{\alpha}{}^2\tilde{L}_i} + \sum_{j=1}^{i=1} h_{j+1}\mu_{\alpha}\tilde{L}_i\right) - \sum_{\beta=1}^{n+1}\left(K\sqrt{\frac{n\sigma_{\alpha}^2}{n+1}\tilde{L}_i} + \sum_{j=1}^{i=1} h_{j+1}\frac{n\mu_{\alpha}}{n+1}\tilde{L}_i\right)$$

$$= n\left(K\sqrt{\sigma_{\alpha}{}^2\tilde{L}_i} + \sum_{j=1}^{i=1} h_{j+1}\mu_{\alpha}\tilde{L}_i\right) - (n+1)\left(K\sqrt{\frac{n\sigma_{\alpha}^2}{n+1}\tilde{L}_i} + \sum_{j=1}^{i=1} h_{j+1}\frac{n\mu_{\alpha}}{n+1}\tilde{L}_i\right)$$

$$= K\left(n\sqrt{\sigma_{\alpha}{}^2\tilde{L}_i} - (n+1)\sqrt{\frac{n\sigma_{\alpha}^2}{n+1}\tilde{L}_i}\right) + n\sum_{j=1}^{i=1} h_{j+1}\mu_{\alpha}\tilde{L}_i - (n+1)\left(\sum_{j=1}^{i=1} h_{j+1}\frac{n\mu_{\alpha}}{n+1}\tilde{L}_i\right)$$

$$
\begin{aligned}
&= K\left(n\sqrt{\sigma_\alpha{}^2 \tilde{L}_i} - (n+1)\sqrt{\frac{n\sigma_\alpha^2}{n+1}\tilde{L}_i} \right) + n\sum_{j=1}^{i=1} h_{j+1}\mu_\alpha \tilde{L}_i - n\left(\sum_{j=1}^{i=1} h_{j+1}\mu_\alpha \tilde{L}_i \right) \\
&= K\left(n\sqrt{\sigma_\alpha{}^2 \tilde{L}_i} - (n+1)\sqrt{\frac{n\sigma_\alpha^2}{n+1}\tilde{L}_i} \right) \\
&= K\left(n\sqrt{\sigma_\alpha{}^2 \tilde{L}_i} - \sqrt{n(n+1)\sigma_\alpha^2 \tilde{L}_i} \right) \\
&= K\left(\sqrt{n^2\sigma_\alpha{}^2 \tilde{L}_i} - \sqrt{n(n+1)\sigma_\alpha^2 \tilde{L}_i} \right) \\
&< 0
\end{aligned}
$$

The proof for $C_i^{u,d}$ follows exactly as above if we instead let $K = \left(b + \sum_{j=1}^{m} h_i \right)\phi\left(z_i^{l,c}\right)$.

For Propositions 4 and 5, we assume the leadtime demand at installation i is uniformly distributed. Specifically, we will consider Uniform(0,1) distributions. Let f(⋅) and F(⋅) represent the Uniform(0,1) pdf and cdf, respectively. The base stock levels become $F^{-1}(\Theta) = \Theta$.

Following the standard approach (e.g. see pp 205-209 in Zipkin (2000); proofs of these derivations are available from the authors upon request)

$$C_{i,\alpha}^{l,d}\left(s_{i,\alpha}^{u,d}\right) = \frac{1}{2}\left(1-\Theta_{i,\alpha}^{u,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha} \right) + \sum_{j=1}^{i-1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i \tag{A14}$$

$$C_{i,\alpha}^{u,d}\left(s_{i,\alpha}^{l,d}\right) = \frac{1}{2}\left(1-\Theta_{i,\alpha}^{l,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha} \right) + \sum_{j=1}^{i-1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i \tag{A15}$$

$$C_{i}^{l,c}\left(s_{i}^{u,c}\right) = \frac{1}{2}\left(1-\Theta_{i}^{u,c}\right)\left(b + \sum_{j=i+1}^{m} h_{j} \right) + \sum_{j=1}^{i-1} h_{j+1}\mu\, \tilde{L}_i \tag{A16}$$

$$C_{i}^{u,c}\left(s_{i}^{l,c}\right) = \frac{1}{2}\left(1-\Theta_{i}^{l,c}\right)\left(b + \sum_{j=i+1}^{m} h_{j} \right) + \sum_{j=1}^{i-1} h_{j+1}\mu\, \tilde{L}_i \tag{A17}$$

Proof of Proposition 4.

Here we investigate the effects of backorder asymmetry. We consider two cases, the collapsed and decomposed systems.

(a) For the collapsed systems, $b = \frac{1}{\mu}\sum_{\alpha=1}^{n} \mu_\alpha b_\alpha$ remains unchanged. Hence the critical fractiles $\Theta_i^{u,c}$ and $\Theta_i^{l,c}$ remain unchanged and hence stocking levels and equations A16 and A17 remain unchanged.

(b) For the decomposed systems, let $B > A > 0$. We will use A and B to denote the sums of the echelon holding costs as is convenient for $\Theta^{l,d}_{i,\alpha}$ and $\Theta^{u,d}_{i,\alpha}$. That is, let $A = \sum_{j=i+1}^{m} h_{j,\alpha}$ and $B = \sum_{j=1}^{m} h_{j,\alpha}$ or $\sum_{j=i}^{m} h_{j,\alpha}$, respectively. Consider

$$
\begin{aligned}
&2s^d_{i,\alpha} - s^d_{i,1} - s^d_{i,2} \\
&= 2F^{-1}\left(\Theta^d_{i,\alpha}\right) - F^{-1}\left(\Theta^d_{i,1}\right) - F^{-1}\left(\Theta^d_{i,2}\right) \\
&= 2F^{-1}\left(\frac{b_\alpha + A}{b_\alpha + B}\right) - F^{-1}\left(\frac{b_1 + A}{b_1 + B}\right) - F^{-1}\left(\frac{b_2 + A}{b_2 + B}\right) \\
&= 2F^{-1}\left(\frac{b + A}{b + B}\right) - F^{-1}\left(\frac{b + A + \Delta}{b + B + \Delta}\right) - F^{-1}\left(\frac{b + A - \Delta}{b + B - \Delta}\right) \\
&= 2\frac{b + A}{b + B} - \frac{b + A + \Delta}{b + B + \Delta} - \frac{b + A - \Delta}{b + B - \Delta}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{2(b+A)(b+B+\Delta)(b+B-\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)} - \frac{(b+B)(b+A+\Delta)(b+B-\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)} \\
- \frac{(b+B)(b+B+\Delta)(b+A-\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)}
\end{aligned}
$$

After expansion and intermediate collection of like terms,

$$
\begin{aligned}
&= \frac{2\left(b^3 + 2b^2B + b^2A + 2bAB + bB^2 + AB^2 - b\Delta^2 - A\Delta^2\right)}{(b+B)(b+B+\Delta)(b+B-\Delta)} \\
&- \frac{b^3 + b^2A + 2b^2B + 2bAB + bB\Delta + bB^2 + AB^2 + B^2\Delta - bA\Delta - b\Delta^2 - AB\Delta - B\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)} \\
&- \frac{b^3 + 2b^2B + bB^2 + b^2A + 2bAB + bA\Delta + AB^2 + AB\Delta - b^2\Delta - bB\Delta - B^2\Delta - B\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)}
\end{aligned}
$$

Which, further simplified is

$$
\frac{2(B-A)\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)}
$$

> 0, and increasing in Δ.

Thus increasing asymmetry in backordering costs decreases equations A2 and A4.

To see the results for the effects on system costs, let $B = \left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right)$. First consider $2C_{i,\alpha}^{u,d}\left(s_{i,\alpha}^{l,d}\right) - C_{i,1}^{u,d}\left(s_{i,2}^{l,d}\right) - C_{i,1}^{u,d}\left(s_{i,2}^{l,d}\right)$

$$= \left(1-\Theta_{i,\alpha}^{l,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{1}{2}\left(1-\Theta_{i,1}^{l,d}\right)\left(b_1 + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{1}{2}\left(1-\Theta_{i,2}^{l,d}\right)\left(b_2 + \sum_{j=i+1}^{m} h_{j,\alpha}\right)$$

$$+2\sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i - \sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L} - \sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i$$

$$= \left(1-\Theta_{i,\alpha}^{l,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{1}{2}\left(1-\Theta_{i,1}^{l,d}\right)\left(b_\alpha + \Delta + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{1}{2}\left(1-\Theta_{i,2}^{l,d}\right)\left(b_\alpha - \Delta + \sum_{j=i+1}^{m} h_{j,\alpha}\right)$$

$$+2\sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i - \sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L} - \sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i$$

$$= \left(1-\Theta_{i,\alpha}^{l,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{1}{2}\left(1-\Theta_{i,1}^{l,d}\right)\left(b_\alpha + \Delta + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{1}{2}\left(1-\Theta_{i,2}^{l,d}\right)\left(b_\alpha - \Delta + \sum_{j=i+1}^{m} h_{j,\alpha}\right)$$

$$= \left(1 - \frac{B}{B + \sum_{j=1}^{i} h_j}\right)B - \frac{1}{2}\left(1 - \frac{B+\Delta}{B+\Delta+\sum_{j=1}^{i} h_j}\right)(B+\Delta) - \frac{1}{2}\left(1 - \frac{B-\Delta}{B-\Delta+\sum_{j=1}^{i} h_j}\right)(B-\Delta)$$

$$= B - \frac{B^2}{B + \sum_{j=1}^{i} h_j} - \frac{(B+\Delta)}{2} + \frac{(B+\Delta)^2}{2\left(B+\Delta+\sum_{j=1}^{i} h_j\right)} - \frac{(B-\Delta)}{2} + \frac{(B-\Delta)^2}{2\left(B-\Delta+\sum_{j=1}^{i} h_j\right)}$$

$$= -\frac{B^2}{B + \sum_{j=1}^{i} h_j} + \frac{(B+\Delta)^2}{2\left(B+\Delta+\sum_{j=1}^{i} h_j\right)} + \frac{(B-\Delta)^2}{2\left(B-\Delta+\sum_{j=1}^{i} h_j\right)}$$

Letting $A = \sum_{j=1}^{i} h_j$

$$= -\frac{B^2}{B+A} + \frac{(B+\Delta)^2}{2(B+\Delta+A)} + \frac{(B-\Delta)^2}{2(B-\Delta+A)}$$

$$= \frac{1}{2}\frac{2B^2(B+\Delta+A)(B-\Delta+A) - (B+A)(B+\Delta)^2(B-\Delta+A) - (B+A)(B-\Delta)^2(B+\Delta+A)}{(B+A)(B+\Delta+A)(B-\Delta+A)}$$

And after expansion and collection of terms,

$$=\frac{\Delta^2 A^2}{(B+A)(B+\Delta+A)(B-\Delta+A)}$$

$>$ 0, and increasing with Δ

Note that the above analysis also holds when $\sum_{j=1}^{i} h_{j,\alpha}$ is replaced by $h_{i,\alpha}$, in which case the proof applies for $2C_{i,\alpha}^{l,d}\left(s_{i,\alpha}^{u,d}\right)-C_{i,1}^{l,d}\left(s_{i,2}^{u,d}\right)-C_{i,1}^{l,d}\left(s_{i,2}^{u,d}\right)$.

Thus asymmetry in backorder cost decreases equations A14 and A15. Combined with the above results, we find that stocking levels and system costs both decrease with increasing asymmetry.

Proof of Proposition 5.

We show the effects of holding cost asymmetry on stocking levels and total system costs. Beginning with the effects on stocking levels, we consider two cases, the collapsed and the decomposed systems.

(a) For the collapsed system, with $h_{f,1} = h_f(1+\Delta)$, and $h_{f,2} = h_f(1-\Delta)$, consider that the weighted holding cost $h_f = \frac{1}{\mu}\left(\sum_{\alpha\neq 1,2} h_{f,\alpha} + \left(h_{f,\alpha}+\Delta\right)+\left(h_{f,\alpha}-\Delta\right)\right)$ is independent of Δ. Hence $\Theta_i^{l,c}$ and $\Theta_i^{u,c}$, and thus the collapsed system stocking levels are independent of Δ.

(b) For the decomposed system, consider $2s_{i,\alpha}^{d}-s_{i,1}^{d}-s_{i,2}^{d}$ where $\alpha \neq 1,2$. We consider five cases, as in Proposition 1b. These are $s_i^{l,d}$ for $1 \leq i < f$ and $i \geq f$, and $s_i^{u,d}$ for $1 \leq i < f$, $i = f$, and $i > f$.

(b1) Consider $s_i^{l,d}$ and $1 \leq i < f$

$$2s_{i,\alpha}^{l,d}-s_{i,1}^{l,d}-s_{i,2}^{l,d}$$
$$=2F^{-1}\left(\Theta_{i,\alpha}^{l,d}\right)-F^{-1}\left(\Theta_{i,1}^{l,d}\right)-F^{-1}\left(\Theta_{i,2}^{l,d}\right)$$

Let $A = \sum_{j=i+1}^{m} h_{j,\alpha}$ and $B = \sum_{j=1}^{m} h_{j,\alpha}$ and note that since $i < f$, h_f affects both A and B.

$$=2F^{-1}\left(\frac{b+A}{b+B}\right)-F^{-1}\left(\frac{b+A+\Delta}{b+B+\Delta}\right)-F^{-1}\left(\frac{b+A-\Delta}{b+B-\Delta}\right)$$
$$=2\frac{b+A}{b+B}-\frac{b+A+\Delta}{b+B+\Delta}-\frac{b+A-\Delta}{b+B-\Delta}$$

These are precisely the terms developed in Proposition 4b. The asymmetry in either holding or backordering cost in echelons below f induces the same behavior. Hence stocking levels decrease with asymmetry in echelon holding costs for $s_i^{l,d}$ and $1 \le i < f$.

(b2) Consider $s_i^{u,d}$ and $1 \le i < f$, and let $A = \sum_{j=i+1}^{m} h_{j,\alpha}$ $B = \sum_{j=i}^{m} h_{j,\alpha}$

$$2s_{i,\alpha}^{u,d}-s_{i,1}^{u,d}-s_{i,2}^{u,d}$$
$$=2F^{-1}\left(\Theta_{i,\alpha}^{u,d}\right)-F^{-1}\left(\Theta_{i,1}^{u,d}\right)-F^{-1}\left(\Theta_{i,2}^{u,d}\right)$$
$$=2F^{-1}\left(\frac{b+A}{b+B}\right)-F^{-1}\left(\frac{b+A+\Delta}{b+B+\Delta}\right)-F^{-1}\left(\frac{b+A-\Delta}{b+B-\Delta}\right)$$
$$=2\frac{b+A}{b+B}-\frac{b+A+\Delta}{b+B+\Delta}-\frac{b+A-\Delta}{b+B-\Delta}$$

Again, the analysis is identical to Proposition 4b.

(b3) Consider $s_i^{l,d}$ and $i \ge f$, and let $A = \sum_{j=i+1}^{m} h_{j,\alpha}$ and $B = \sum_{j=1}^{m} h_{j,\alpha}$

$$2s_{i,\alpha}^{l,d}-s_{i,1}^{l,d}-s_{i,2}^{l,d}$$
$$=2F^{-1}\left(\Theta_{i,\alpha}^{l,d}\right)-F^{-1}\left(\Theta_{i,1}^{l,d}\right)-F^{-1}\left(\Theta_{i,2}^{l,d}\right)$$
$$=2F^{-1}\left(\frac{b+A}{b+B}\right)-F^{-1}\left(\frac{b+A}{b+B+\Delta}\right)-F^{-1}\left(\frac{b+A}{b+B-\Delta}\right)$$
$$=2\frac{b+A}{b+B}-\frac{b+A}{b+B+\Delta}-\frac{b+A}{b+B-\Delta}$$

$$=\frac{2(b+A)(b+B+\Delta)(b+B-\Delta)-(b+A)(b+B)(b+B-\Delta)-(b+A)(b+B)(b+B+\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)}$$

Expanding the terms yields

$$=\frac{2\left(b^3+2b^2B+b^2A+2bAB+bB^2+AB^2-b\Delta^2-A\Delta^2\right)}{(b+B)(b+B+\Delta)(b+B-\Delta)}$$

$$\frac{-\left(b^3+2b^2B+b^2A+2bAB+bB^2+AB^2-b^2\Delta-bB\Delta-bA\Delta-AB\Delta\right)}{(b+B)(b+B+\Delta)(b+B-\Delta)}$$

$$\frac{-\left(b^3+2b^2B+b^2A+2bAB+bB^2+AB^2+b^2\Delta+bB\Delta+bA\Delta+AB\Delta\right)}{(b+B)(b+B+\Delta)(b+B-\Delta)}$$

Cancelling terms yields

$$=\frac{-2(b+A)\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)}$$

< 0, and decreasing in Δ

(b4) Consider $s_i^{u,d}$ and $i = f$, and let $A = \sum_{j=i+1}^{m} h_{j,\alpha}$ $B = \sum_{j=i}^{m} h_{j,\alpha}$

$$\begin{aligned}
&2s_{i,\alpha}^{u,d}-s_{i,1}^{u,d}-s_{i,2}^{u,d}\\
&=2F^{-1}\left(\Theta_{i,\alpha}^{l,d}\right)-F^{-1}\left(\Theta_{i,1}^{l,d}\right)-F^{-1}\left(\Theta_{i,2}^{l,d}\right)\\
&=2F^{-1}\left(\frac{b+A}{b+B}\right)-F^{-1}\left(\frac{b+A}{b+B+\Delta}\right)-F^{-1}\left(\frac{b+A}{b+B-\Delta}\right)\\
&=2\frac{b+A}{b+B}-\frac{b+A}{b+B+\Delta}-\frac{b+A}{b+B-\Delta}
\end{aligned}$$

Which proceeds precisely as the above proof in Proposition 5b3. Hence $s_i^{u,d}$ decreases with increasing Δ when $i = f$.

(b5) Consider $s_i^{u,d}$ and $i > f$. Inspection of $\Theta_i^{u,d} = \frac{b+\sum_{j=i+1}^{m} h_j}{b+\sum_{j=i}^{m} h_j}$ shows that $\Theta_i^{u,d}$ is independent of h_f when $i > f$. Hence $2s_{i,\alpha}^{u,d}-s_{i,1}^{u,d}-s_{i,2}^{u,d} = 0$.

Combining Propositions 5b1 through 5b5 results in the finding that increasing asymmetry in h_f decreases the system stocking levels downstream of the point of asymmetry, while increasing stocking levels at or upstream of the point of asymmetry in the decomposed system. Because the collapsed system is insulated from the effects of h_f, $s_i^a = \frac{s_i^c + s_i^d}{2}$ behaves in the same manner.

For the effects of holding cost asymmetry on total system costs, we again consider the two cases of the collapsed and decomposed systems.

(c) For the collapsed system, consider that $h_f = \frac{1}{\mu}\sum_{\alpha=1}^{n}\mu_\alpha h_{f,\alpha}$ remains unchanged.

Then equations A16 and A17,

$$C_i^{l,c}\left(s_i^{u,c}\right) = \frac{1}{2}\left(1-\Theta_i^{u,c}\right)\left(b+\sum_{j=i+1}^{m}h_j\right)+\sum_{j=1}^{i-1}h_{j+1}\mu\,\tilde{L}_i \text{ and}$$

$$C_i^{u,c}\left(s_i^{l,c}\right) = \frac{1}{2}\left(1-\Theta_i^{l,c}\right)\left(b+\sum_{j=i+1}^{m}h_j\right)+\sum_{j=1}^{i-1}h_{j+1}\mu\,\tilde{L}_i,$$

also remain unchanged (recall that Proposition 5 states that $\Theta_i^{u,c}$ and $\Theta_i^{l,c}$ are independent from asymmetry in h_f.)

(d) For the decomposed system, we consider two cases, i < f and $i \geq f$.

(d1) For $i \geq f$, first consider $2C_{i,\alpha}^{u,d} - C_{i,1}^{u,d} - C_{i,2}^{u,d}$ and let

$A = \left(\sum_{j=1}^{i}h_{j,\alpha}\right)$ and $B = \left(b_\alpha + \sum_{j=i+1}^{m}h_{j,\alpha}\right)$.

$$=\left(1-\Theta_{i,\alpha}^{l,d}\right)\left(b_\alpha+\sum_{j=i+1}^{m}h_{j,\alpha}\right)+\sum_{j=1}^{i=1}h_{j+1,\alpha}\mu_\alpha\tilde{L}_i$$

$$-\frac{1}{2}\left(1-\Theta_{i,1}^{l,d}\right)\left(b_1+\sum_{j=i+1}^{m}h_{j,1}\right)+\sum_{j=1}^{i=1}h_{j+1,1}\mu_1\tilde{L}_i$$

$$-\frac{1}{2}\left(1-\Theta_{i,2}^{l,d}\right)\left(b_2+\sum_{j=i+1}^{m}h_{j,2}\right)+\sum_{j=1}^{i=1}h_{j+1,2}\mu_2\tilde{L}_i$$

$$=\left(1-\Theta_{i,\alpha}^{l,d}\right)\left(b_\alpha+\sum_{j=i+1}^{m}h_{j,\alpha}\right)-\frac{1}{2}\left(1-\Theta_{i,1}^{l,d}\right)\left(b_1+\sum_{j=i+1}^{m}h_{j,1}\right)-\frac{1}{2}\left(1-\Theta_{i,2}^{l,d}\right)\left(b_2+\sum_{j=i+1}^{m}h_{j,2}\right)$$

$$=B\left(1-\frac{B}{B+A}\right)-\frac{B}{2}\left(1-\frac{B}{B+A+\Delta}\right)-\frac{B}{2}\left(1-\frac{B}{B+A-\Delta}\right)$$

$$=B-\frac{B^2}{B+A}-\frac{B}{2}+\frac{1}{2}\frac{B^2}{B+\Delta+A}-\frac{B}{2}+\frac{1}{2}\frac{B^2}{B+A-\Delta}$$

$$=-\frac{B^2}{B+A}+\frac{B^2}{2\left(B+\Delta+A\right)}+\frac{B^2}{2\left(B+A-\Delta\right)}$$

$$= \frac{B^2\left(2(B+A)(B-\Delta+A)+2(B+A)(B+\Delta+A)-4(B+\Delta+A)(B+A-\Delta)\right)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

$$= \frac{B^2\left(\left(B^2-B\Delta+AB+AB-A\Delta+A^2\right)+\left(B^2+B\Delta+AB+AB+A\Delta+A^2\right)\right)}{2(B+A)(B+\Delta+A)(B+A-\Delta)}$$
$$-\frac{2B^2\left(B^2+AB-B\Delta+AB+A^2-A\Delta+B\Delta+A\Delta-\Delta^2\right)}{2(B+A)(B+\Delta+A)(B+A-\Delta)}$$

Collecting like terms gives

$$= \frac{-B^2\Delta^2}{(B+A)(B+\Delta+A)(B+A-\Delta)}$$

< 0, and decreasing in Δ.

For $C_i^{l,d}$ consider $2C_{i,\alpha}^{u,d} - C_{i,1}^{u,d} - C_{i,2}^{u,d}$

$$= \left(1-\Theta_{i,\alpha}^{u,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) + \sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i$$
$$- \frac{\left(1-\Theta_{i,1}^{u,d}\right)}{2}\left(b_1 + \sum_{j=i+1}^{m} h_{j,1}\right) + \sum_{j=1}^{i=1} h_{j+1,1}\mu_1 \tilde{L}_i$$
$$- \frac{\left(1-\Theta_{i,2}^{u,d}\right)}{2}\left(b_2 + \sum_{j=i+1}^{m} h_{j,2}\right) + \sum_{j=1}^{i=1} h_{j+1,2}\mu_2 \tilde{L}_i$$

$$= \left(1-\Theta_{i,\alpha}^{u,d}\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{\left(1-\Theta_{i,1}^{u,d}\right)}{2}\left(b_1 + \sum_{j=i+1}^{m} h_{j,1}\right) - \frac{\left(1-\Theta_{i,2}^{u,d}\right)}{2}\left(b_2 + \sum_{j=i+1}^{m} h_{j,2}\right)$$
$$= B\left(1-\frac{B}{B+h_{i,\alpha}}\right) - \frac{B}{2}\left(1-\frac{B}{B+h_{i,1}}\right) - \frac{B}{2}\left(1-\frac{B}{B+h_{i,2}}\right)$$

$$= -\frac{B^2}{B+h_{i,\alpha}} + \frac{B^2}{2\left(B+h_{i,1}\right)} + \frac{B^2}{2\left(B+h_{i,2}\right)}$$

For $i > f$, $-\frac{B^2}{B+h_{i,\alpha}} + \frac{B^2}{2\left(B+h_{i,1}\right)} + \frac{B^2}{2\left(B+h_{i,2}\right)} = 0$,

$$= -\frac{B^2}{B+h_{i,\alpha}} + \frac{B^2}{2\left(B+h_{i,1}\right)} + \frac{B^2}{2\left(B+h_{i,2}\right)}$$

Letting A = $h_{i,\alpha}$

$$= -\frac{B^2}{B+A} + \frac{B^2}{2(B+\Delta+A)} + \frac{B^2}{2(B+A-\Delta)}$$

$$= \frac{B^2\left(2(B+A)(B-\Delta+A)+2(B+A)(B+\Delta+A)-4(B+\Delta+A)(B+A-\Delta)\right)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

$$= \frac{B^2\left(\left(B^2-B\Delta+AB+AB-A\Delta+A^2\right)+\left(B^2+B\Delta+AB+AB+A\Delta+A^2\right)\right)}{2(B+A)(B+\Delta+A)(B+A-\Delta)}$$
$$-\frac{2B^2\left(B^2+AB-B\Delta+AB+A^2-A\Delta+B\Delta+A\Delta-\Delta^2\right)}{2(B+A)(B+\Delta+A)(B+A-\Delta)}$$

and collecting like terms gives

$$= \frac{-B^2\Delta^2}{(B+A)(B+\Delta+A)(B+A-\Delta)}$$

< 0, and decreasing in Δ.

(d2) For $i < f$, let $A =$ $h_{i,\alpha}$ or $\left(\sum_{j=1}^{i} h_{j,\alpha}\right)$ for C_i^l and C_i^u, respectively, and let $B = \left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right)$.

Consider $2C_{i,\alpha}^d - C_{i,1}^d - C_{i,2}^d$

$$= \left(1-\Theta_{i,\alpha}^d\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) + \sum_{j=1}^{i=1} h_{j+1,\alpha}\mu_\alpha \tilde{L}_i$$
$$-\frac{\left(1-\Theta_{i,1}^d\right)}{2}\left(b_1 + \sum_{j=i+1}^{m} h_{j,1}\right) + \sum_{j=1}^{i=1} h_{j+1,1}\mu_1 \tilde{L}_i$$
$$-\frac{\left(1-\Theta_{i,2}^d\right)}{2}\left(b_2 + \sum_{j=i+1}^{m} h_{j,2}\right) + \sum_{j=1}^{i=1} h_{j+1,2}\mu_2 \tilde{L}_i$$

$$= \left(1-\Theta_{i,\alpha}^d\right)\left(b_\alpha + \sum_{j=i+1}^{m} h_{j,\alpha}\right) - \frac{\left(1-\Theta_{i,1}^d\right)}{2}\left(b_1 + \sum_{j=i+1}^{m} h_{j,1}\right) - \frac{\left(1-\Theta_{i,2}^d\right)}{2}\left(b_2 + \sum_{j=i+1}^{m} h_{j,2}\right)$$

$$= B\left(1-\frac{B}{B+A}\right)-\frac{(B+\Delta)}{2}\left(1-\frac{B+\Delta}{B+A+\Delta}\right)-\frac{(B-\Delta)}{2}\left(1-\frac{B-\Delta}{B+A-\Delta}\right)$$

$$= B-\frac{B^2}{B+A}-\frac{B}{2}+\frac{1}{2}\frac{(B+\Delta)^2}{B+\Delta+A}-\frac{B}{2}+\frac{1}{2}\frac{(B-\Delta)^2}{B+A-\Delta}$$

$$= -\frac{B^2}{B+A}+\frac{(B+\Delta)^2}{2(B+\Delta+A)}+\frac{(B-\Delta)^2}{2(B+A-\Delta)}$$

$$= \frac{2(B+\Delta)^2(B+A-\Delta)(B+A)+2(B-\Delta)^2(B+A+\Delta)(B+A)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$
$$-\frac{4B^2(B+A-\Delta)(B+A+\Delta)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

Expanding the terms yields

$$= \frac{2\left(A^2B^2+2AB^3+B^4+2A^2B\Delta+3AB^2\Delta+B^3\Delta+A^2\Delta^2-B^2\Delta^2-A\Delta^3-B\Delta^3\right)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

$$+\frac{2\left(A^2B^2+2AB^3+B^4-2A^2B\Delta-3AB^2\Delta-B^3\Delta+A^2\Delta^2-B^2\Delta^2+A\Delta^3+B\Delta^3\right)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

$$-\frac{4\left(A^2B^2+2AB^3+B^4-B^2\Delta^2\right)}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

Collecting like terms gives

$$= \frac{2A^2\Delta^2}{4(B+A)(B+\Delta+A)(B+A-\Delta)}$$

> 0, and increasing in Δ.

Hence increasing asymmetry in echelon *f* holding costs increases the costs incurred downstream of echelon *f*, but decreases the costs of operating the system from the viewpoint of installations at and upstream of *f*, including *m*, and hence the total system costs.

We now consider asymmetry in demand. Because demand does not appear in the critical fractiles, we may revert to normal distributions.

Proof of Proposition 6.

We consider two cases, the collapsed and decomposed systems.

(a) For the collapsed system, note that $\mu = \sum_{\alpha=1}^{n} \mu_\alpha$ and $\sigma^2 = \sum_{\alpha=1}^{n} \sigma_\alpha^2$ are independent of asymmetry in μ_α and σ_α^2. Then the collapsed stocking levels

$s_i^{l,c} = \mu\ \tilde{L}_i + z_i^{l,c}\sqrt{\sigma^2 \tilde{L}_i}$ and $s_i^{u,c} = \mu\ \tilde{L}_i + z_i^{u,c}\sqrt{\sigma^2 \tilde{L}_i}$

are likewise independent of asymmetry in μ_α and σ_α^2. Also, the cost equations

$$C_i^{l,c}\left(s_i^{u,c}\right) = \left(b + \sum_{j=i}^{m} h_i\right)\phi\left(z_i^{u,c}\right)\sqrt{\sigma^2 \tilde{L}_i} + \sum_{j=1}^{i-1} h_{j+1}\mu\ \tilde{L}_i$$

and $C_i^{u,c}\left(s_i^{l,c}\right) = \left(b + \sum_{j=1}^{m} h_i\right)\phi\left(z_i^{l,c}\right)\sqrt{\sigma^2 \tilde{L}_i} + \sum_{j=1}^{i-1} h_{j+1}\mu\ \tilde{L}_i$

are likewise independent of asymmetry in μ_α and σ_α^2.

(b) For the decomposed system, first consider 2 $s_{i,\alpha}{}^{d}$- $s_{i,1}{}^{d}$- $s_{i,2}{}^{d}$

$$= z_i^d \sqrt{\tilde{L}_i}\left(2\sqrt{\sigma_\alpha^2} - \sqrt{\frac{\mu+\Delta}{\mu}\sigma_\alpha^2} - \sqrt{\frac{\mu-\Delta}{\mu}\sigma_\alpha^2}\right)$$

$$= z_i^d \sqrt{\tilde{L}_i \sigma_\alpha^2}\left(2 - \sqrt{\frac{\mu+\Delta}{\mu}} - \sqrt{\frac{\mu-\Delta}{\mu}}\right)$$

$$\le z_i^d \sqrt{\tilde{L}_i \sigma_\alpha^2}\left(2 - \sqrt{\frac{\mu}{\mu}} - \sqrt{\frac{\Delta}{\mu}} - \sqrt{\frac{\mu}{\mu}} + \sqrt{\frac{\Delta}{\mu}}\right)$$

= 0.

Next, consider $2C_{i,\beta}^{u} - C_{i,1}^{u} - C_{i,2}^{u}$

$$= 2\left(b + \sum_{j=1}^{m} h_j\right)\phi\left(z_i^l\right)\sqrt{\sigma_\alpha^2 \tilde{L}_i} - \left(b + \sum_{j=1}^{m} h_j\right)\phi\left(z_i^l\right)\sqrt{\sigma_1^2 \tilde{L}_i} - \left(b + \sum_{j=1}^{m} h_j\right)\phi\left(z_i^l\right)\sqrt{\sigma_2^2 \tilde{L}_i}$$

$$= \left(b + \sum_{j=1}^{m} h_j\right)\phi\left(z_i^l\right)\sqrt{\tilde{L}_i}\left(2\sqrt{\sigma_\alpha^2} - \sqrt{\sigma_1^2} - \sqrt{\sigma_2^2}\right)$$

$$= \left(b + \sum_{j=1}^{m} h_j\right)\phi\left(z_i^l\right)\sqrt{\tilde{L}_i}\left(2\sqrt{\sigma_\alpha^2} - \sqrt{\frac{\mu+\Delta}{\mu}\sigma_\alpha^2} - \sqrt{\frac{\mu-\Delta}{\mu}\sigma_\alpha^2}\right)$$

$$\leq \left(b + \sum_{j=1}^{m} h_j \right) \phi\left(z_i^l\right) \sqrt{\widetilde{L}_i \sigma_\alpha^2} \left(2 - \sqrt{\frac{\mu}{\mu}} - \sqrt{\frac{\Delta}{\mu}} - \sqrt{\frac{\mu}{\mu}} + \sqrt{\frac{\Delta}{\mu}} \right)$$

$= 0.$

Also, $2C_{i,\alpha}^l - C_{i,1}^l - C_{i,2}^l$ follows as above, substituting $\sum_{j=i}^{m} h_j$ for $\sum_{j=1}^{m} h_j$ and $\phi\left(z_i^u\right)$ for $\phi\left(z_i^l\right)$

Hence for the decomposed chains, asymmetry in demand processes decreases both stocking levels and system costs. This result carries over to the arborescent system because of the independence (and thus non-interference) of the collapsed system.